第 八 章

波色统计和费米统计

§8.1 热力学量的统计表达

一、从非简并到简并

玻耳兹曼系统（玻耳兹曼分布）　　孤立系统

定域粒子组成的系统，满足经典极限条件（非简并条件）的近独立粒子系统

经典极限条件（非简并条件）

$$a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}\ \mathrm{ml}}$$

$$e^{\alpha} >> 1$$

$$a_l = \omega_l e^{-\alpha-\beta\varepsilon_l}$$

玻色分布和费米分布 趋向于玻耳兹曼分布。

$$Z_1 = \sum_{l=0}^{\infty} \omega_l e^{-\beta\varepsilon_l} = \sum_{l=0}^{\infty} \frac{a_l}{e^{-a}}$$

$$e^{-\alpha} = \frac{N}{Z_1}$$

$$Z_1 = V\left(\frac{2\pi m}{h^2\beta}\right)^{3/2}$$

$$e^{\alpha} = \frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2} >> 1$$

$$e^{-\alpha} = \frac{N}{V}\left(\frac{h^2}{2\pi mkT}\right)^{3/2} = n\lambda^3 << 1$$

不满足非简并条件

开放系统，与源达到动态平衡，粒子数在能级上的平均分布。

采用玻色分布或费米分布

$$a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l} \pm 1}$$

费米统计

玻色统计

二、巨配分函数

$$\overline{N} = \sum_l a_l = \sum_l \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l} \pm 1}$$

$$U = \sum_l \varepsilon_l a_l = \sum_l \frac{\varepsilon_l \omega_l}{e^{\alpha+\beta\varepsilon_l} \pm 1}$$

$$\Xi = \prod_l \Xi_l = \prod_l (1 \pm e^{-\alpha-\beta\varepsilon_l})^{\pm\omega_l}$$

$$\ln \Xi = \pm \sum_l \omega_l \ln(1 \pm e^{-\alpha-\beta\varepsilon_l})$$

对比玻耳兹曼分布

$$Z_1 = \sum_{l=0}^{\infty} \omega_l e^{-\beta\varepsilon_l}$$

三、用巨配分函数表示热力学量

1 平均粒子数 $\overline{N}$

$$\overline{N}=\sum_l a_l=\sum_l \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}\pm 1}$$

$$\ln\Xi=\pm\sum_l \omega_l \ln(1\pm e^{-\alpha-\beta\varepsilon_l})$$

$$-\frac{\partial}{\partial\alpha}\ln\Xi=\mathrm{m}\frac{\partial}{\partial\alpha}\sum_l \omega_l \ln(1\pm e^{-\alpha-\beta\varepsilon_l})$$

$$=\mathrm{m}\sum_l \omega_l \frac{\pm e^{-\alpha-\beta\varepsilon_l}(-1)}{1\pm e^{-\alpha-\beta\varepsilon_l}}$$

$$=\sum_l \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}\pm 1} \quad =\overline{N}$$

$$\overline{N}=-\frac{\partial}{\partial\alpha}\ln\Xi$$

对比玻耳兹曼分布

$$N=Z_1 e^{-\alpha}$$

2 内能

$$U=\sum_l \varepsilon_l a_l=\sum_l \frac{\varepsilon_l\omega_l}{e^{\alpha+\beta\varepsilon_l}\pm 1}$$

$$\ln\Xi=\pm\sum_l \omega_l \ln(1\pm e^{-\alpha-\beta\varepsilon_l})$$

$$-\frac{\partial}{\partial\beta}\ln\Xi=\mathrm{m}\frac{\partial}{\partial\beta}\sum_l \omega_l \ln(1\pm e^{-\alpha-\beta\varepsilon_l})$$

$$=\mathrm{m}\sum_l \omega_l \frac{\pm e^{-\alpha-\beta\varepsilon_l}(-\varepsilon_l)}{1\pm e^{-\alpha-\beta\varepsilon_l}}$$

$$=\sum_l \frac{\omega_l\varepsilon_l}{e^{\alpha+\beta\varepsilon_l}\pm 1} \quad =U$$

$$U=-\frac{\partial}{\partial\beta}\ln\Xi$$

对比玻耳兹曼分布

$$U=-N\frac{\partial \ln Z_1}{\partial\beta}$$

3 广义力

$$Y=\sum_l a_l \frac{\partial \varepsilon_l}{\partial y}$$

$$\ln \Xi = \pm \sum_l \omega_l \ln(1 \pm e^{-\alpha-\beta\varepsilon_l})$$

$$-\frac{1}{\beta}\frac{\partial}{\partial y}\ln \Xi = \mathrm{m}\frac{1}{\beta}\frac{\partial}{\partial y}\sum_l \omega_l \ln(1 \pm e^{-\alpha-\beta\varepsilon_l})$$

$$= \mathrm{m}\sum_l \omega_l \frac{\pm e^{-\alpha-\beta\varepsilon_l}(-1)}{1 \pm e^{-\alpha-\beta\varepsilon_l}}\frac{\partial \varepsilon_l}{\partial y} = \sum_l \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l} \pm 1}\frac{\partial \varepsilon_l}{\partial y} = \sum_l a_l \frac{\partial \varepsilon_l}{\partial y} = Y$$

对比玻耳兹曼分布

$$Y = -\frac{1}{\beta}\frac{\partial}{\partial y}\ln \Xi$$

$$Y = -N\frac{1}{\beta}\frac{\partial \ln Z_1}{\partial y}$$

压强

$$p = \frac{1}{\beta}\frac{\partial}{\partial V}\ln \Xi$$

$$p = \frac{N}{\beta}\frac{\partial \ln Z_1}{\partial V}$$

4 其它热力学函数

由开系的热力学公式 $dU - Ydy - \mu dN = TdS$

$$\beta(dU - Ydy + \frac{\alpha}{\beta} d\overline{N}) = -\beta d(\frac{\partial}{\partial \beta}\ln \Xi) + \frac{\partial}{\partial y}\ln \Xi dy - \alpha d(\frac{\partial}{\partial \alpha}\ln \Xi)$$

$$= -d(\beta \frac{\partial}{\partial \beta}\ln \Xi) + \frac{\partial}{\partial \beta}\overset{*}{\ln} \Xi d\beta + \frac{\partial}{\partial y}\overset{*}{\ln} \Xi dy - d(\alpha \frac{\partial}{\partial \alpha}\ln \Xi) + \frac{\partial}{\partial \alpha}\overset{*}{\ln} \Xi d\alpha$$

$$= -d(\beta \frac{\partial}{\partial \beta}\ln \Xi) - d(\alpha \frac{\partial}{\partial \alpha}\ln \Xi) + d(\overset{*}{\ln} \Xi)$$

$$= d(\ln \Xi - \beta \frac{\partial}{\partial \beta}\ln \Xi - \alpha \frac{\partial}{\partial \alpha}\ln \Xi)$$

$$= \beta TdS$$

$$\beta(dU - Ydy + \frac{\alpha}{\beta} d\overline{N}) = d(\ln\Xi - \beta\frac{\partial}{\partial\beta}\ln\Xi - \alpha\frac{\partial}{\partial\alpha}\ln\Xi) = \beta T dS$$

$$\Rightarrow \quad \beta = \frac{1}{kT} \qquad \alpha = -\frac{\mu}{kT}$$

熵

$$dS = kd(\ln\Xi - \beta\frac{\partial}{\partial\beta}\ln\Xi - \alpha\frac{\partial}{\partial\alpha}\ln\Xi)$$

$$S = k(\ln\Xi - \beta\frac{\partial}{\partial\beta}\ln\Xi - \alpha\frac{\partial}{\partial\alpha}\ln\Xi)$$

$$U = -\frac{\partial}{\partial\beta}\ln\Xi$$

$$S = k(\ln\Xi + \beta U + \alpha\overline{N})$$

$$\overline{N} = -\frac{\partial}{\partial\alpha}\ln\Xi$$

与玻耳兹曼关系比较

$$S = k\ln\Omega$$

对于玻色分布

$$\Omega_{B.E} = \prod_l \frac{(\omega_l + a_l - 1)!}{a_l!(\omega_l - 1)!}$$

$$\ln\Omega_{B.E} = \sum_l (\omega_l + a_l)\ln(\omega_l + a_l) - \sum_l \omega_l \ln\omega_l - \sum_l a_l \ln a_l$$

$$S = k\ln\Omega_{B.E} = k(\sum_l (\omega_l + a_l)\ln(\omega_l + a_l) - \sum_l \omega_l \ln\omega_l - \sum_l a_l \ln a_l)$$

$$S = k(\ln\Xi + \beta U + \alpha N)$$

? $$= k\ln\Omega_{B.E} = k(\sum_l (\omega_l + a_l)\ln(\omega_l + a_l) - \sum_l \omega_l \ln\omega_l - \sum_l a_l \ln a_l)$$

$$a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l} - 1} \rightarrow e^{\alpha+\beta\varepsilon_l} = \frac{\omega_l + a_l}{a_l} \rightarrow \alpha + \beta\varepsilon_l = \ln\frac{\omega_l + a_l}{a_l}$$

$$\downarrow$$

$$1 - e^{-\alpha-\beta\varepsilon_l} = \frac{\omega_l}{\omega_l + a_l}$$

$$\ln\Xi = -\sum_l \omega_l \ln(1 - e^{-\alpha-\beta\varepsilon_l}) \quad = -\sum \omega_l \ln\frac{\omega_l}{\omega_l + a_l}$$

$$U = \sum \varepsilon_l a_l \qquad N = \sum a_l$$

$$\beta U + \alpha N = \sum \beta\varepsilon_l a_l + \sum a a_l = \sum a_l(a + \beta\varepsilon_l) = \sum a_l \ln(\frac{\omega_l + a_l}{a_l})$$

$$S = k(\ln\Xi + \beta U + \alpha N) = k(-\sum \omega_l \ln\frac{\omega_l}{\omega_l + a_l} + \sum a_l \ln(\frac{\omega_l + a_l}{a_l})) \quad = k\ln\Omega_{B.E}$$

对于费米分布

$$\Omega_{F.D}=\prod_l \frac{\omega_l!}{a_l!(\omega_l-a_l)!}$$

$$\ln\Omega_{F.D}=\sum_l \omega_l\ln\omega_l-\sum_l a_l\ln a_l-\sum_l(\omega_l-a_l)\ln(\omega_l-a_l)$$

$$S=k\ln\Omega_{F.D}=k(\sum_l \omega_l\ln\omega_l-\sum_l a_l\ln a_l-\sum_l(\omega_l-a_l)\ln(\omega_l-a_l))$$

$$S=k(\ln\Xi+\beta U+\alpha N)$$

? $$=k\ln\Omega_{F.D}=k(\sum_l \omega_l\ln\omega_l-\sum_l a_l\ln a_l-\sum_l(\omega_l-a_l)\ln(\omega_l-a_l))$$

$$a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}+1} \Rightarrow e^{\alpha+\beta\varepsilon_l} = \frac{\omega_l - a_l}{a_l} \Rightarrow \alpha + \beta\varepsilon_l = \ln\frac{\omega_l - a_l}{a_l}$$

$$1 + e^{-\alpha-\beta\varepsilon_l} = \frac{\omega_l}{\omega_l - a_l}$$

$$\ln\Xi = \sum_l \omega_l \ln(1 + e^{-\alpha-\beta\varepsilon_l}) \qquad = \sum \omega_l \ln\frac{\omega_l}{\omega_l - a_l}$$

$$U = \sum \varepsilon_l a_l \qquad N = \sum a_l$$

$$\beta U + \alpha N = \sum \beta\varepsilon_l a_l + \sum a a_l = \sum a_l (a + \beta\varepsilon_l) = \sum a_l \ln(\frac{\omega_l - a_l}{a_l})$$

$$S = k(\ln\Xi + \beta U + \alpha N) = k(-\sum \omega_l \ln\frac{\omega_l}{\omega_l - a_l} + \sum a_l \ln(\frac{\omega_l - a_l}{a_l})) \qquad = k\ln\Omega_{F.D}$$

§8.2 弱简并玻色气体和费米气体

玻色统计与费米统计描述不可区分的粒子系统。主要是空间中不可区分。但当粒子在空间可以区分时（稀薄气体），应该由描述可区分粒子系统的理论－玻耳兹曼统计－描述。

$$a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l} \mp 1}$$

$e^{\alpha} >> 1$

$$a_l = \omega_l e^{-\alpha-\beta\varepsilon_l}$$

$e^{-\alpha}$虽小但不可忽略

一、 弱简并气体

$$\frac{1}{e^{\alpha+\beta\varepsilon_l} \pm 1} = \frac{1}{e^{\alpha+\beta\varepsilon_l}(1 \pm e^{-\alpha-\beta\varepsilon_l})}$$

$$\frac{1}{1 \pm e^{-\alpha-\beta\varepsilon_l}} \approx 1 \mp e^{-\alpha-\beta\varepsilon_l}$$

$$\frac{1}{1+e^{-x}} = 1 - e^{-x} + e^{-2x} - \cdots$$

$$a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l} \pm 1} \approx \omega_l e^{-\alpha-\beta\varepsilon_l}(1 \mp e^{-\alpha-\beta\varepsilon_l})$$

考虑平动 $\varepsilon = \frac{\vec{p}^2}{2m}$

粒子微观状态数 $D(\varepsilon)d\varepsilon = g\frac{2\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2}d\varepsilon$ 6.2.17式

总粒子数

$$N = \int_0^\infty D(\varepsilon)a(\varepsilon)d\varepsilon = g\frac{2\pi V}{h^3}(2m)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}d\varepsilon}{e^{\alpha+\beta\varepsilon_l}\pm 1}$$

$$= g\frac{2\pi V}{h^3}(2m)^{3/2}\int_0^\infty \varepsilon^{1/2}e^{-\alpha-\beta\varepsilon_l}(1\mp e^{-\alpha-\beta\varepsilon_l})d\varepsilon$$

$$= g\frac{2\pi V}{h^3}(2m)^{3/2}e^{-\alpha}(\int_0^\infty \varepsilon^{1/2}e^{-\beta\varepsilon_l}d\varepsilon \mp \int_0^\infty \varepsilon^{1/2}e^{-\alpha-2\beta\varepsilon_l}d\varepsilon)$$

$$\Rightarrow N = g(\frac{2\pi mkT}{h^2})^{3/2}Ve^{-\alpha}(1\mp\frac{1}{2^{3/2}}e^{-\alpha})$$

内能

$$U=\int_0^\infty D(\varepsilon)a(\varepsilon)\varepsilon d\varepsilon=\frac{3}{2}g\left(\frac{2\pi mkT}{h^2}\right)^{3/2}Ve^{-\alpha}kT\left(1\mp\frac{1}{2^{5/2}}e^{-\alpha}\right)$$

又

$$N=g\left(\frac{2\pi mkT}{h^2}\right)^{3/2}Ve^{-\alpha}\left(1\mp\frac{1}{2^{3/2}}e^{-\alpha}\right)$$

两式相除得到

$$U\approx\frac{3}{2}NkT\left(1\pm\frac{1}{2^{5/2}}e^{-\alpha}\right)$$

近似求解过程：

$$N = g\frac{2\pi V}{h^3}(2m)^{3/2}e^{-\alpha}\left(\int_0^{\infty}\varepsilon^{1/2}e^{-\beta\varepsilon_l}d\varepsilon \mp \int_0^{\infty}\varepsilon^{1/2}e^{-\alpha-2\beta\varepsilon_l}d\varepsilon\right)$$

$$\int_0^{\infty}\varepsilon^{1/2}e^{-\beta\varepsilon_l}d\varepsilon$$

令$t = \beta\varepsilon$,

则$\varepsilon^{1/2} = \frac{1}{\beta^{1/2}}t^{1/2}, d\varepsilon = \frac{1}{\beta}dt$

$$\int_0^{\infty}\varepsilon^{1/2}e^{-\beta\varepsilon_l}d\varepsilon = \frac{1}{\beta^{3/2}}\int_0^{\infty}t^{1/2}\cdot e^{-t}dt$$

附录 C.15

$$= \frac{2}{\beta^{3/2}}\int_0^{\infty}y^2\cdot e^{-y^2}dy = \frac{2}{\beta^{3/2}}\cdot\frac{\sqrt{\pi}}{4} = \frac{\sqrt{\pi}}{2\beta^{3/2}}$$

$$\int_0^{\infty}\varepsilon^{1/2}e^{-\alpha-2\beta\varepsilon_l}d\varepsilon = e^{-\alpha}\int_0^{\infty}\varepsilon^{1/2}e^{-2\beta\varepsilon}d\varepsilon$$

$$= e^{-\alpha}\frac{\sqrt{\pi}}{2(2\beta)^{3/2}}$$

$$g\frac{2\pi V}{h^3}(2m)^{3/2}e^{-\alpha}\left(\int_0^{\infty}\varepsilon^{1/2}e^{-\beta\varepsilon_l}d\varepsilon \mp \int_0^{\infty}\varepsilon^{1/2}e^{-\alpha-2\beta\varepsilon_l}d\varepsilon\right)$$

$$= g\frac{2\pi V}{h^3}(2m)^{3/2}e^{-\alpha}\frac{\sqrt{\pi}}{2\beta^{3/2}}\left(1 \mp \frac{e^{-\alpha}}{2^{3/2}}\right) = g\left(\frac{2\pi mkT}{h^2}\right)^{3/2}Ve^{-\alpha}\left(1 \mp \frac{e^{-\alpha}}{2^{3/2}}\right)$$

$$N = g\left(\frac{2\pi mkT}{h^2}\right)^{3/2} V e^{-\alpha}\left(1 \mp \frac{1}{2^{3/2}} e^{-\alpha}\right)$$

$$U = \frac{3}{2} g\left(\frac{2\pi mkT}{h^2}\right)^{3/2} V e^{-\alpha} kT\left(1 \mp \frac{1}{2^{5/2}} e^{-\alpha}\right)$$

令 $\frac{1}{2^{3/2}} = e^{-x}$

$$\frac{1}{1+e^{-x}} = 1 - e^{-x} + e^{-2x} - \cdots$$

$$\frac{1+\frac{1}{2^{5/2}}e^{-\alpha}}{1+\frac{1}{2^{3/2}}e^{-\alpha}} = \frac{1+\frac{1}{2}e^{-\alpha-x}}{1+e^{-\alpha-x}} = \left(1+\frac{1}{2}e^{-\alpha-x}\right)\left(1-e^{-\alpha-x}\right)$$

$$= 1 - \frac{1}{2}e^{-\alpha-x} + \frac{1}{2}e^{-2\alpha-2x} = 1 - \frac{1}{2^{5/2}}e^{-\alpha}$$

$$U = \frac{3}{2} NkT\left(1 \pm \frac{1}{2^{5/2}} e^{-\alpha}\right)$$

二、 弱简并条件物理含义

利用玻耳兹曼统计的结果

$$e^{-\alpha} = \frac{N}{Z_1} = \frac{N}{V}\frac{1}{g}\left(\frac{h^2}{2\pi mkT}\right)^{3/2} \ll 1$$

$$U = \frac{3}{2}NkT\left(1 \pm \frac{1}{2^{5/2}}\frac{N}{V}\frac{1}{g}\left(\frac{h^2}{2\pi mkT}\right)^{3/2}\right)$$

第一项：根据玻耳兹曼分布得到的内能

第二项：微观粒子全同性引起的量子统计关联导致的附加内能

费米粒子相互排斥；玻色粒子相互吸引。

§8.3 玻色—爱因斯坦凝聚

一、玻色气体的化学势

玻色分布下一个能级的粒子数

$$a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1} = \frac{\omega_l}{e^{\frac{\varepsilon_l-\mu}{kT}}-1}$$

$$\alpha = -\frac{\mu}{kT}$$

$$0 \le a_l = \frac{\omega_l}{e^{\frac{\varepsilon_l-\mu}{kT}}-1}$$

$$e^{\frac{\varepsilon_l-\mu}{kT}} > 1$$

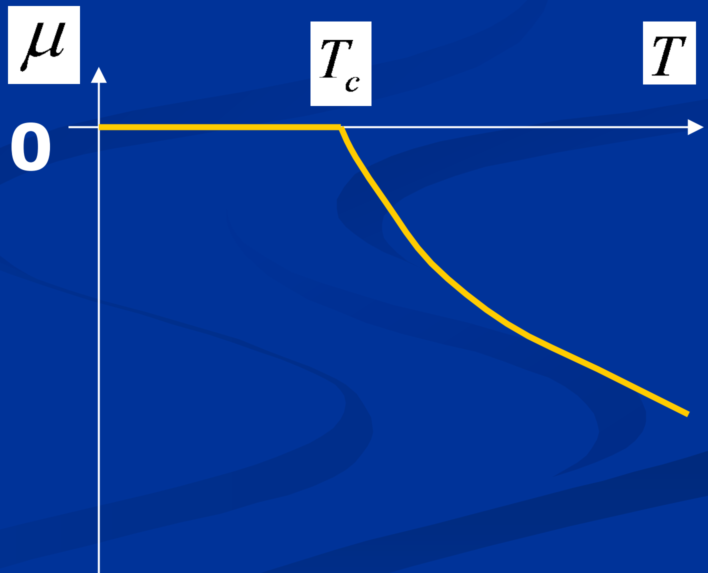
$$\varepsilon_0 > \mu$$

最低能级 $\varepsilon_0 = 0$

$$\Rightarrow \mu(N,T,V) < 0$$

在粒子数给定情况下，μ 与T的关系

$$n = \frac{N}{V} = \frac{1}{V}\sum_l \frac{\omega_l}{e^{\frac{\varepsilon_l-\mu}{kT}}-1}$$



μ 随温度的升高而降低

$$n=\frac{N}{V}=\frac{1}{V}\sum_l\frac{\omega_l}{e^{\frac{\varepsilon_l-\mu}{kT}}-1}$$

$$D(\varepsilon)d\varepsilon=\frac{2\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2}d\varepsilon.$$

连续化

$$n=\frac{N}{V}=\frac{1}{V}\int_0^\infty\frac{D(\varepsilon)}{\omega_l}a_l d\varepsilon=\frac{1}{V}\int_0^\infty D(\varepsilon)a(\varepsilon)d\varepsilon$$

$$n=n_0+\frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty\frac{\varepsilon^{1/2}d\varepsilon}{e^{\frac{\varepsilon-\mu}{kT}}-1}$$

$\varepsilon=0$ 能级　　$\varepsilon>0$ 能级

$T>T_c$　n_0 可以忽略

$$n=\frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty\frac{\varepsilon^{1/2}d\varepsilon}{e^{\frac{\varepsilon+|\mu|}{kT}}-1}$$

$$n=\frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty\frac{\varepsilon^{1/2}d\varepsilon}{e^{\frac{\varepsilon}{kT_c}}-1}$$

临界温度T_c：所有玻色粒子都在非零能级的最低温度

$T<T_c$

n_0可以和所有激发态能级上粒子数相比较，即粒子都往 $\varepsilon=0$ 能级聚集。

令 $x = \frac{\varepsilon}{kT_c}$

$$n = \frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}d\varepsilon}{e^{\frac{\varepsilon}{kT_c}}-1}$$

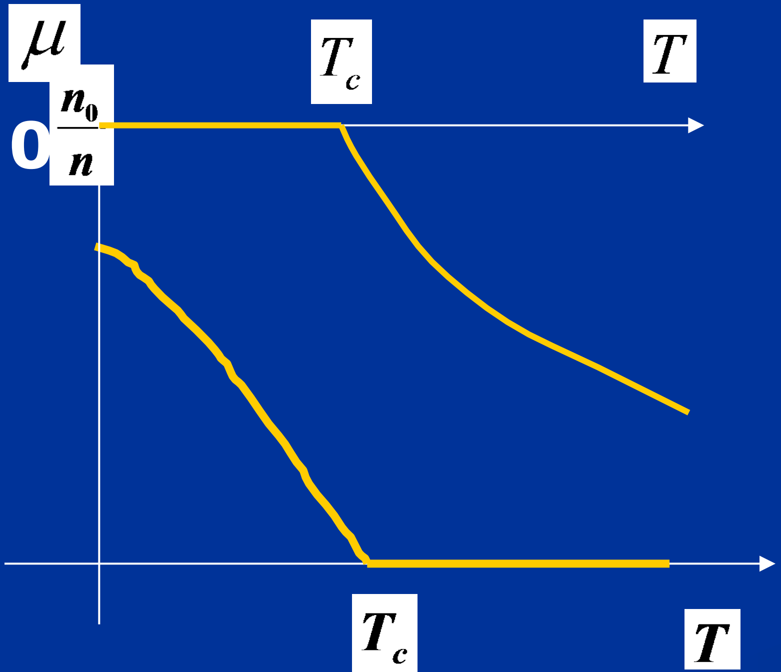
$$= \frac{2\pi}{h^3}(2mkT_c)^{3/2}\int_0^\infty \frac{x^{1/2}dx}{e^x-1}$$

$$\int_0^\infty \frac{x^{1/2}dx}{e^x-1} = \frac{\sqrt{\pi}}{2}\times 2.612$$

$$T_c = \frac{2\pi}{(2.612)^{2/3}}\frac{h^2}{mk}n^{2/3}$$

$T < T_c, \mu = 0$

$\varepsilon > 0$



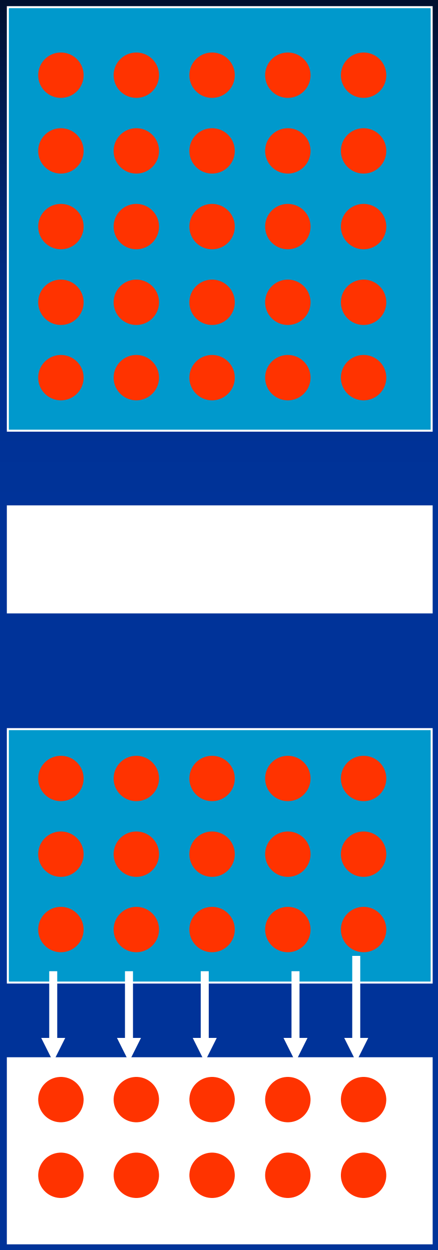
$$n_{\varepsilon>0} = \frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}d\varepsilon}{e^{\frac{\varepsilon}{kT}}-1}$$

$$= \frac{2\pi}{h^3}(2mkT)^{3/2}\int_0^\infty \frac{x^{1/2}dx}{e^x-1}$$

$$= \left(\frac{T}{T_c}\right)^{3/2}\frac{2\pi}{h^3}(2mkT_c)^{3/2}\int_0^\infty \frac{x^{1/2}dx}{e^x-1}$$

$$= n\left(\frac{T}{T_c}\right)^{3/2}$$

$$n_0 = n\left[1-\left(\frac{T}{T_c}\right)^{3/2}\right]$$



$T > T_c$

$\varepsilon > 0$

$$n_0 = n[1 - (\frac{T}{T_c})^{3/2}] \approx 0$$

$$n_{\varepsilon>0} = n(\frac{T}{T_c})^{3/2} \approx n$$

$\varepsilon = 0$

玻色粒子都在高能级。

$T < T_c$

$\varepsilon > 0$

$$n_0 = n[1 - (\frac{T}{T_c})^{3/2}] \neq 0$$

$$n_{\varepsilon>0} = n(\frac{T}{T_c})^{3/2} < n$$

$\varepsilon = 0$

高能级装不下所有玻色粒子，
必有可观数目粒子出现在零能级。
——玻色—爱因斯坦凝聚。

$$T_c = \frac{2\pi}{(2.612)^{2/3}} \frac{h^2}{mk} n^{2/3}$$

因此，为了容易实现玻色—爱因斯坦凝聚，需要提高临界温度。

为此，要提高气体密度，减小气体粒子质量。

二、热力学量

T<T $_c$时

$$n = \frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}d\varepsilon}{e^{\frac{\varepsilon}{kT}}-1}$$

$$U = \frac{2\pi}{h^3}(2m)^{3/2}\int_0^\infty \frac{\varepsilon^{3/2}d\varepsilon}{e^{\frac{\varepsilon}{kT}}-1}$$

$$= 0.770\,NkT\left(\frac{T}{T_c}\right)^{3/2}$$

$$C_V = 1.925\,Nk\left(\frac{T}{T_c}\right)^{3/2}$$

T<T$_c$，理想玻色气体的C$_V$与T$^{3/2}$成正比，T=T $_c$达极大值。 高温时趋于经典值 $\frac{3}{2}Nk$

三、发展过程

1. 理论准备

1924.6.24，印度人玻色给爱因斯坦寄“玻色分布”文章。

经爱因斯坦努力，该论文发表。

在这篇文章基础上，爱因斯坦继续发表论文，提出“玻色凝聚”Bose-Einstein Condensation (BEC) 的概念。

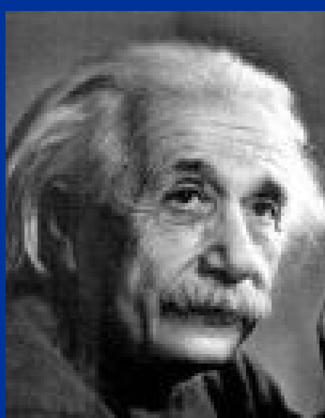
2. 实验检验

1995年7月13日，美国科罗拉多大学报告：铷 (^{87}Rb) 蒸气在170nK出现BEC。

8月，休斯顿Rice大学宣布，在锂 (^{7}Li) 蒸气中出现BEC。

11月，麻省理工学院宣布，钠 (^{23}Na) 蒸气中出现BEC.

S. Bose



A. Einstein

1924年，玻色和爱因斯坦在理论上预言了玻色—爱因斯坦凝聚（BEC:Bose-Einstein Condensation）现象, 如果将原子气体冷却到非常低的温度，那么所有原子会突然以可能的最低能态凝聚。

§8.4 光子气体

一、光子气体特性

光子——辐射场能量的量子化，自旋 1—玻色子。

平衡辐射场中，光子数不守恒。

空窖壁不断吸收和发射光子，保持能量守恒，但光子能量有高有低，发射光子平均能量高发射光子数目少，被吸收的光子平均能量低，被吸收的光子数目就多，因此不要求光子数守恒。

光子气体服从玻色分布

化学势描述物质变化

$$a_l = \frac{\omega_l}{e^{\alpha+\beta\varepsilon_l}-1} = \frac{\omega_l}{e^{\frac{\varepsilon_l-\mu}{kT}}-1} = \frac{\omega_l}{e^{\hbar\omega/kT}-1}$$

$$\mu = 0$$

二、普朗克公式

德布罗意关系：$\vec{p} = \hbar \vec{k},\quad \varepsilon = \hbar\omega.$

色散关系：$k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi f}{c} = \omega / c$

分布：$a_l = \dfrac{\omega_l}{e^{\hbar\omega/kT} - 1}$

光子能动关系 $\varepsilon = cp$

动量空间 $p - p + dp$ 中量子态数

$$g\frac{dxdydzdp_x dp_y dp_z}{h^3} = 2\frac{V4\pi}{h^3}p^2dp$$

$$= \frac{8\pi V}{h^3}p^2dp$$

频率空间 $\omega - \omega + d\omega$ 中量子态数

$$= \frac{V}{\pi^2 c^3}\omega^2 d\omega$$

$d\omega \to 0$

$\leftrightarrow \omega_l$

一个量子态的平均粒子数 $f=\frac{a_l}{\omega_l}=\frac{1}{e^{\hbar\omega/kT}-1}$

频率空间 $\omega-\omega+d\omega$ 中平均光子数

$$f\times\frac{V\omega^2 d\omega}{\pi^2c^3}=\frac{V}{\pi^2c^3}\frac{\omega^2 d\omega}{e^{\hbar\omega/kT}-1}$$

普朗克公式

(辐射场内能)

$$U(\omega,T)d\omega=\frac{V}{\pi^2c^3}\frac{\omega^2 d\omega}{e^{\hbar\omega/kT}-1}\times\hbar\omega=\frac{V}{\pi^2c^3}\frac{\hbar\omega^3 d\omega}{e^{\hbar\omega/kT}-1}$$

低频（弱简并），经典描述——能量均分定理。

$\hbar\omega/kT<<1$ $e^{\hbar\omega/kT}\approx 1+\hbar\omega/kT$

$$U(\omega,T)d\omega\approx\frac{V}{\pi^2c^3}\frac{\hbar\omega^3 d\omega}{1+\hbar\omega/kT-1}=\frac{V}{\pi^2c^3}\omega^2 kTd\omega$$

瑞利-金斯公式

高频 $\hbar\omega / kT >> 1$

$$U(\omega,T)d\omega = \frac{V}{\pi^2 c^3}\hbar\omega^3 e^{-\hbar\omega/kT}d\omega$$

U随ω的增加迅速趋近于零。温度为T的平衡辐射中，高频光子几乎不存在；温度为T时，窖壁发射高频光子的概率极小。

三、平衡辐射公式

$x = \hbar\omega / kT$

1. 内能

$$U = \frac{V}{\pi^2 c^3}\int_0^\infty \frac{\hbar\omega^3 d\omega}{e^{\hbar\omega/kT}-1} = \frac{V\hbar}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^4\int_0^\infty \frac{\hbar x^3 dx}{e^x - 1}$$

$$U = \frac{\pi^2 k^4}{15\hbar^3 c^3}VT^4$$

$U = aVT^4$

$$a = \frac{\pi^2 k^4}{15\hbar^3 c^3}$$

热力学只能通过实验确定系数a；统计物理可以计算a

2. 维恩位移律

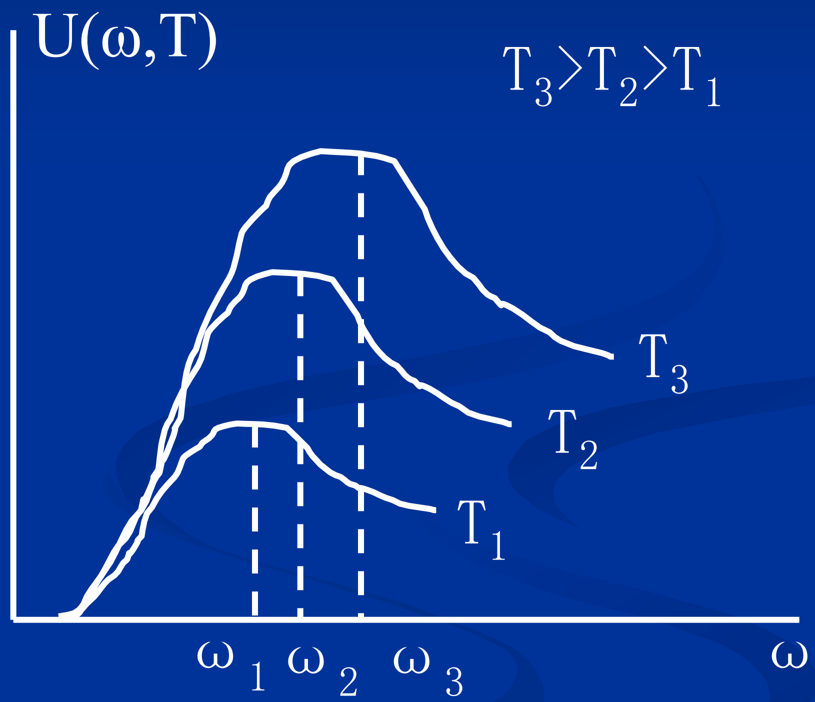
内能最大的频率 ω_m

$$\frac{\partial}{\partial\omega}U(\omega,T)\Big|_{\omega=\omega_m}=0$$

$$\frac{\partial}{\partial x}\frac{x^3}{e^x-1}=\frac{3x^2}{e^x-1}-\frac{x^3e^x}{(e^x-1)^2}=0$$

$$3e^x-3-xe^x=0$$

$$\omega_m\approx\frac{2.822k}{\hbar}T$$



3. 压强、辐射通量密度

$\omega - \omega + d\omega$ 中量子态数 $\frac{V}{\pi^2 c^3}\omega^2 d\omega \leftrightarrow \omega_l$

$x = \hbar\omega / kT$

$$\ln \Xi = -\sum_l \omega_l \ln(1 - e^{-\alpha-\beta\varepsilon_l}) = -\frac{V}{\pi^2 c^3}\int_0^\infty \omega^2 \ln(1 - e^{-\beta\hbar\omega}) d\omega$$

$$\ln \Xi = -\frac{V}{\pi^2 c^3}\frac{1}{(\beta\hbar)^3}\int_0^\infty x^2 \ln(1 - e^{-x}) dx$$

分部积分

$$\int_0^\infty x^2 \ln(1 - e^{-x}) dx = \left[\frac{x^3}{3}\ln(1 - e^{-x})\right]_0^\infty - \frac{1}{3}\int_0^\infty \frac{x^3 dx}{e^x - 1}$$

$$\ln \Xi = \frac{V}{3\pi^2 c^3}\frac{1}{(\beta\hbar)^3}\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^2 V}{45 c^3}\frac{1}{(\beta\hbar)^3}$$

$$\ln \Xi = \frac{\pi^2 V}{45c^3}\frac{1}{(\beta \mathrm{h})^3}$$

$$U = -\frac{\partial}{\partial \beta}\ln \Xi = \frac{\pi^2 k^4 V}{15c^3 \mathrm{h}^3}T^4$$

$$p = \frac{1}{\beta}\frac{\partial}{\partial V}\ln \Xi = \frac{\pi^2 k^4}{45c^3 \mathrm{h}^3}T^4$$

$$p = \frac{1}{3}\frac{U}{V}$$

习题7.2结果

$$S = k\left(\ln \Xi - \beta\frac{\partial}{\partial \beta}\ln \Xi\right) = k(\ln \Xi + \beta U) = \frac{4\pi^2 k^4 V}{45c^3 \mathrm{h}^3}T^3$$

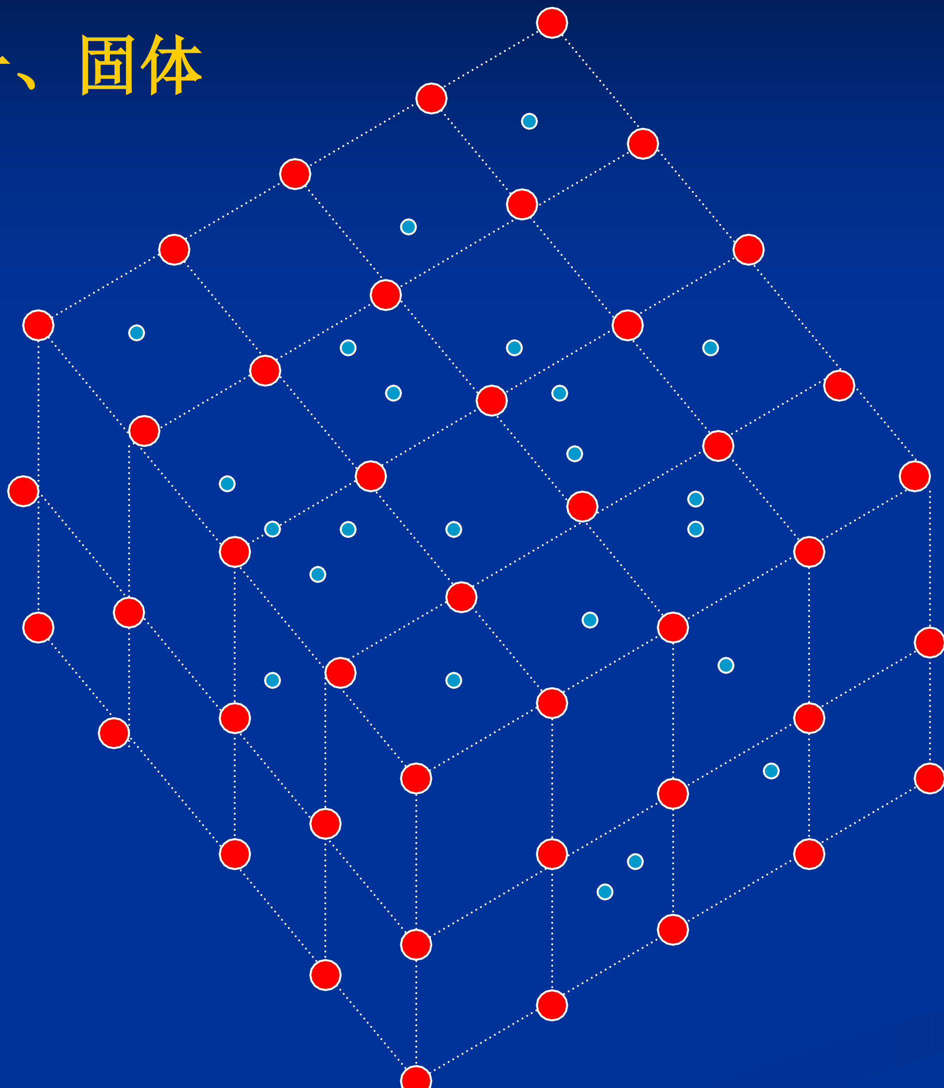
平衡辐射通量密度

$$J_\mu = \frac{c}{4}u = \frac{c}{4}\frac{U}{V} = \frac{\pi^2 k^4}{60c^2 \mathrm{h}^3}T^4$$

强简并理想费米气体作为一种理想化的模型，可以近似描写金属中的巡游电子气体，以及中子星、白矮星和核物质等.

§8.5 金属中的自由电子气体

一、固体



每个原子贡献一个电子，晶格中的自由电子气体。

晶格——三维线性振子

$$U = 3NkT$$

$$C_V = 3Nk$$

电子对热容量的贡献未计！

量子性质

例Cu：$密度 = 8.9\,g.cm^{3}$

$原子量 = 63$

$$n = \frac{8.9\times10^{6}}{63}\times N_0 = 8.5\times10^{28}\,m^{-3}$$

$$m = 9.1\times10^{-31}\,kg$$

$$e^{-\alpha} = n\lambda^{3} = \frac{N}{V}\left(\frac{h^{2}}{2\pi mkT}\right)^{3/2} = \frac{3.54\times10^{7}}{T^{3/2}} = 3400\ ?\ \ 1$$

$n\lambda^{3} = 1$ 非简并条件 → 弱简并 → 强简并