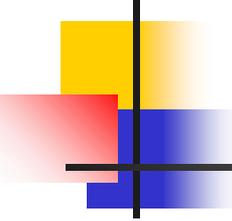


关于数值计算方法三次样条插 值

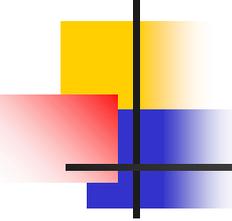




4.4.1 分段插值

已知 $(x_j, y_j), j = 0, 1, \dots, n$, 判断 $x \in [x_{j-1}, x_j]$
则 $f(x)$ 用 $[x_{j-1}, x_j]$ 上的线性插值函数表示

计算机上实现 $u \leftarrow x$, , 若 $u < x_1$ 取 $j = 1$,
即 $x \in [x_0, x_1]$, (若 $u \leq x_0$, 也选 $j = 1$, 则外插
若 $u \leq x_2$, 取 $j = 2$



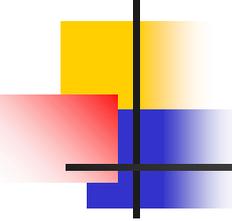
分段线性插值

一般的, $x_{j-1} \leq u \leq x_j$, 则线性插值函数为

$$u = y_{j-1} + (u - x_{j-1})(y_j - y_{j-1}) / (x_j - x_{j-1})$$

这是因为

$$\begin{aligned} y &= \frac{x - x_j}{x_{j-1} - x_j} y_{j-1} + \frac{x - x_{j-1}}{x_j - x_{j-1}} y_j \\ &= y_{j-1} + (x - x_{j-1})(y_j - y_{j-1}) / (x_j - x_{j-1}) \end{aligned}$$



分段线性插值

算法:

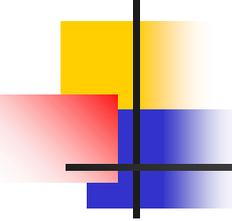
1. 输入 $x_i, y_i (i = 0, 1, \dots, n)$

2. 按 $k = 1, 2, \dots, m$ 做

(1) 输入插值点

(2) 对于 $j = 1, 2, \dots, n$ 做

如果 $u \leq x_j$ 则



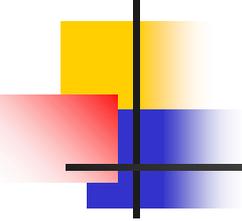
分段线性插值

$$1^0 \quad v = y_{j-1} + (u - x_{j-1})(y_j - y_{j-1}) / (x_j - x_{j-1})$$

2⁰ 输出 u, v

分段插值函数

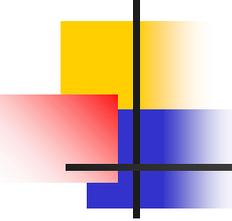
$$I(x) = \begin{cases} I_1(x) & x \in (x_0, x_1) \\ I_2(x) & x \in (x_1, x_2) \\ \dots\dots & \\ I_n(x) & x \in (x_{n-1}, x_n) \end{cases}$$



$$\begin{aligned} \text{其中 } I_j &= \frac{x - x_j}{x_{j-1} - x_j} y_{j-1} + \frac{x - x_{j-1}}{x_j - x_{j-1}} y_j \\ &= y_{j-1} + (x - x_{j-1})(y_j - y_{j-1}) / (x_j - x_{j-1}) \end{aligned}$$

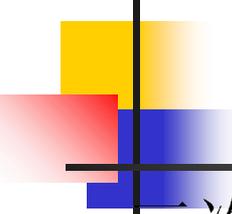
■ 缺点：I(x)连续，但不光滑，精度较低，仅在

$h = \max_{1 \leq j \leq n} \{h_j = x_j - x_{j-1}\}$ 足够小才能较好的逼近



分段三次Hermite插值

- 上述分段线性插值曲线是折线，光滑性差，如果交通工具用这样的外形，则势必加大摩擦系数，增加阻力，因此用hermite分段插值更好。



分段三次Hermite插值

三次Hermite插值 $x \in [x_{j-1}, x_j]$ 时

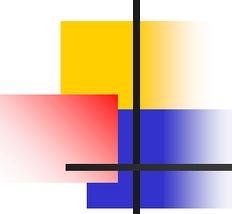
$$H_3(x) = \alpha_{j-1}(x)y_{j-1} + \alpha_j(x)y_j + \beta_{j-1}(x)f'_{j-1} + \beta_j(x)f'_j$$

令 $A_1 = \alpha_{j-1}(u) = \left(1 + 2\frac{u - x_{j-1}}{h_j}\right)\left(\frac{u - x_j}{h_j}\right)^2$

$$A_2 = \alpha_j(u) = \left(1 + 2\frac{u - x_j}{h_j}\right)\left(\frac{u - x_{j-1}}{h_j}\right)^2$$

$$B_1 = \beta_{j-1}(u) = (u - x_{j-1})\left(\frac{u - x_j}{h_j}\right)^2$$

$$B_2 = \beta_j(u) = (u - x_j)\left(\frac{u - x_{j-1}}{h_j}\right)^2$$



分段三次Hermite插值算法

则 $v = A_1 y_{j-1} + A_2 y_j + B_1 f'_{j-1} + B_2 f'_j$

算法:

1. 输入 x_j, f_j, f'_j ($j = 0, 1, \dots, n$) ;

2. 计算插值

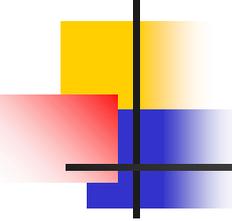
(1) 输入插值点 u ;

(2) 对于 $j = 1, 2, \dots, n$ 做

如果 $u \leq x_j$ 则计算 A_1, A_2, B_1, B_2 ;

$v = A_1 f_{j-1} + A_2 f_j + B_1 f'_{j-1} + B_2 f'_j$;

3. 输出 u, v 。



例题

例 设 $f(1) = 2$, $f(2) = 3$, $f'(1) = 1$, $f'(2) = -1$,
求满足条件的Hermite插值多项式。

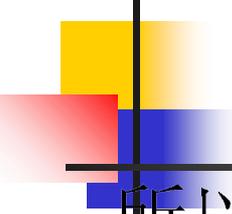
解: $x_0 = 1, x_1 = 2, h = 2 - 1 = 1$ 则

$$A_1 = (1 + 2(x - 1))(x - 2)^2 = (2x - 1)(x - 2)^2$$

$$A_2 = (1 + 2(x - 2))(x - 1)^2 = (2x - 3)(x - 1)^2$$

$$B_1 = (x - 1)(x - 2)^2$$

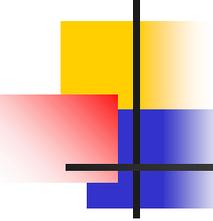
$$B_2 = (x - 2)(x - 1)^2$$



例题

所以得

$$\begin{aligned}H_3(x) &= 2(2x-1)(x-2)^2 + 3(2x-3)(x-1)^2 \\ &= (x-1)(x-2)^2 - (x-2)(x-1)^2 \\ &= -3x^3 + 8x^2 - 9x + 5\end{aligned}$$



4.4.2 三次样条插值

定义 设函数 $f(x)$ 是区间 $[a, b]$ 上的二次连续可微函数
在区间 $[a, b]$ 上给出一个划分

$$\Delta: a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$

如果函数 $s(x)$ 满足条件

- (1) $s(x_j) = f(x_j) \quad (j = 0, 1, 2, \dots, n)$;
 - (2) 在每个小区间 $[x_{j-1}, x_j] (j = 1, 2, \dots, n)$ 上 $s(x)$ 是不超过三次多项式;
 - (3) 在开区间 (a, b) 上 $s(x)$ 有连续的二阶导数
- 则称 $s(x)$ 为区间 $[a, b]$ 对应于划分 Δ 的三次样条函数。

三次样条插值

设三次样条函数 $s(x)$ 在每个子区间 $[x_{j-1}, x_j]$ 上有表达式

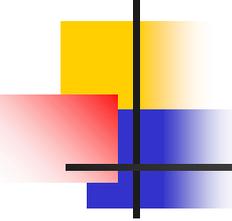
$$s(x) = s_j(x) = a_j x^3 + b_j x^2 + c_j x + d_j \quad x \in (x_{j-1}, x_j), j = 1, 2, \dots, n$$

其中 a_j, b_j, c_j, d_j 为待定常数，插值条件：

(1) $s(x_j) = f(x_j) \quad (j = 0, 1, \dots, n)$;

(2) $(n-1)$ 内节点处连续及光滑条件：

$$\left. \begin{aligned} s(x_j - 0) &= s(x_j + 0) \\ s'(x_j - 0) &= s'(x_j + 0) \\ s''(x_j - 0) &= s''(x_j + 0) \end{aligned} \right\} j = 1, 2, \dots, n-1$$

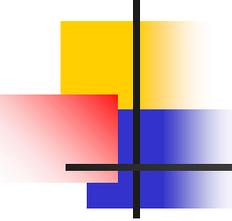


三次样条插值

对于待定系数 $a_j, b_j, c_j, d_j \quad j = 1, 2, \dots, n$, 即 $4n$ 个未知系数
而插值条件为 $n-2$ 个, 还缺两个, 因此给出两个
条件称为边界条件, 分下三类:

第一类 已知两端点的一阶导数

$$\begin{cases} s'(x_0) = f'(x_0) = m_0 \\ s'(x_n) = f'(x_n) = m_n \end{cases}$$



三次样条插值

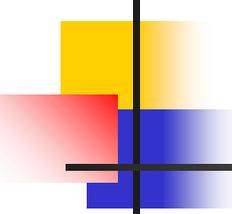
第二类已知两端点二阶导数

$$\begin{cases} s''(x_0) = f''(x_0) = M_0 \\ s''(x_n) = f''(x_n) = M_n \end{cases}$$

当 $M_0 = M_n = 0$ 时为自然边界条件

第三类：周期边界条件

$$\begin{cases} s(x_0) = s(x_n) \\ s'(x_0 + 0) = s'(x_0 - 0) \\ s''(x_n = 0) = s''(x_n - 0) \end{cases}$$

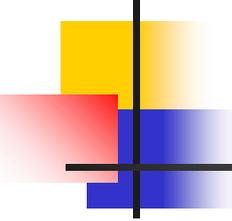


三次样条插值

用三弯矩阵构造三次样条插值函数

令 $s''(x_i) = M_i (i = 0, 1, 2, \dots, n)$ 。因为 $s(x)$ 在 $[x_i, x_{i+1}]$ 上是三次多项式，所以 $s(x)$ 在 $[x_i, x_{i+1}]$ 上是一次多项式，故有

$$s'''(x) = \frac{M_{i+1} - M_i}{x_{i+1} - x_i} \quad \forall x \in [x_i, x_{i+1}]$$



三次样条插值

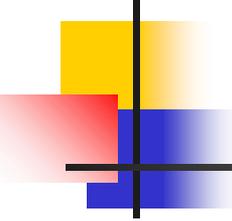
于是由Taylor展示有

$$\begin{aligned} s(x) &= s(x_i) + s'(x_i)(x - x_i) + \frac{s''(x_i)}{2!}(x - x_i)^2 + \frac{s'''(x_i)}{3!}(x - x_i)^3 \\ &= y_i + s'(x_i)(x - x_j) + \frac{M_i}{2!}(x - x_i)^2 + \frac{M_{i+1} - M_i}{3!(x_{i+1} - x_i)}(x - x_i)^3 \end{aligned}$$

令 $x = x_{i+1}$ 得

$$y_{i+1} = y_i + s'(x_i)(x_{i+1} - x_i) + \frac{M_i}{2!}(x_{i+1} - x_i)^2 + \frac{M_{i+1} - M_i}{3!}(x_{i+1} - x_i)^2$$

$$\text{解得 } s'(x_i) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \left(\frac{1}{6}M_{i+1} + \frac{2}{6}M_i\right)(x_{i+1} - x_i) \quad (1)$$



三次样条插值

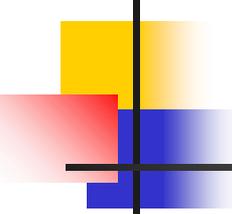
同理在 $[x_{i-1}, x_i]$ 上讨论得

$$s'(x_i) = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} + \left(\frac{2}{6}M_i + \frac{1}{6}M_{i-1}\right)(x_i - x_{i-1}) \quad (2)$$

因为 $s'(x)$ 连续, 所以由(2)即

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \left(\frac{1}{6}M_{i+1} + \frac{2}{6}M_i\right)(x_{i+1} - x_i) = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} + \left(\frac{2}{6}M_i + \frac{1}{6}M_{i-1}\right)(x_i - x_{i-1})$$

$$\text{记 } h_i = x_i - x_{i-1} \quad \mu_i = \frac{h_i}{h_{i+1} + h_i} \quad \lambda_i = 1 - \mu_i = \frac{h_{i+1}}{h_{i+1} + h_i}$$



三次样条插值

则上式为

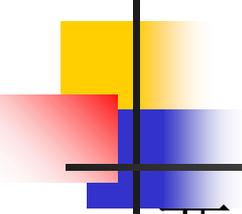
$$f[x_{i+1}, x_i] - \frac{1}{6}(M_{i+1} + 2M_i)h_{i+1} = f[x_i, x_{i-1}] + \frac{1}{6}(2M_i + M_{i-1})h_i$$

即

$$(2M_i + M_{i-1})h_i + (M_{i+1} + 2M_i)h_{i+1} = 6(f[x_{i+1}, x_i] - f[x_i, x_{i-1}])$$

也就是

$$h_i M_{i-1} + 2(h_i + h_{i+1})M_i + h_{i+1}M_{i+1} = 6(f[x_{i+1}, x_i] - f[x_i, x_{i-1}])$$



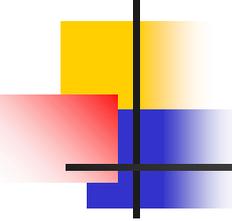
两边同除

$$x_{i+1} - x_{i-1} = (x_{i+1} - x_i + x_i - x_{i-1}) = h_{i+1} + h_i$$

得 $\frac{h_i}{h_{i+1} + h_i} M_{i-1} + 2M_i + \frac{h_i}{h_{i+1} + h_i} M_{i+1} = 6f[x_{i-1}, x_i, x_{i+1}]$

即得

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = 6f[x_{i-1}, x_i, x_{i+1}] \quad i = 1, 2, \dots, n-1$$



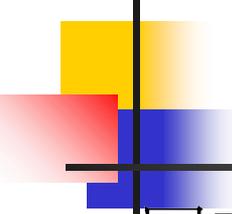
三次样条插值

第一类边界条件: $s'(x_0) = f'(x_0)$ $s'(x_n) = f'(x_n)$

(1) 式中令 $i = 0$ 得

$$s'(x_0) = \frac{y_1 - y_0}{x_1 - x_0} - \left(\frac{1}{6} M_1 + \frac{2}{6} M_0 \right) (x_1 - x_0)$$

既有 $2M_0 + M_1 = 6f[x_0, x_0, x_1]$



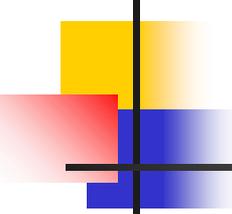
三次样条插值

同理(2)式中令 $i = n$ 得

$$M_{n-1} + 2M_n = 6f[x_{n-1}, x_n, x_n]$$

即有

$$\begin{cases} 2M_0 + M_1 = 6f[x_0, x_0, x_1] \\ \mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = 6f[x_{i-1}, x_i, x_{i+1}] & (i = 1, 2, \dots, n-1) \\ M_{n-1} + 2M_n = 6f[x_{n-1}, x_n, x_n] \end{cases}$$

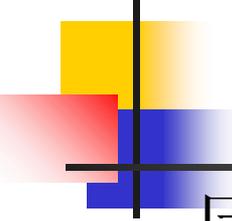


三次样条插值

第二类边界条件 $s''(x_0) = f''(x_0) = M_0, s''(x_n) = f''(x_n) = M_n$

同理可得

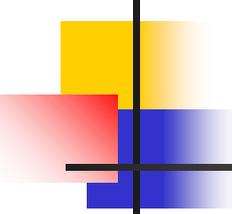
$$\begin{cases} 2M_1 + \lambda_1 M_2 = 6f[x_0, x_1, x_2] - \mu_1 M_0 \\ \mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = 6f[x_{i-1}, x_i, x_{i+1}] \\ \mu_{n-1} M_{n-2} + 2M_n = 6f[x_{n-2}, x_{n-1}, x_n] - \lambda_{n-1} M_n \end{cases}$$
$$i = 2, 3, \dots, n-2$$



三次样条插值

周期函数边界条件下的弯方程

$$\begin{cases} 2M_1 + \lambda_1 M_2 + \mu_1 M_n = f[x_0, x_1, x_2] \\ \mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = 6f[x_{i-1}, x_i, x_{i+1}] \\ \lambda_n M_1 + \mu_n M_{n-1} + 2M_n = 6f[x_{n-1}, x_n, x_{n+1}] \end{cases}$$
$$i = 2, 3, \dots, n-1$$



例题

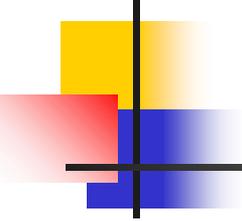
- 例4.4.1 已知函数 $y=f(x)$ 的数表如下表所示。

x	0	0.15	0.30	0.45	0.60
$f(x)$	1	0.97800	0.91743	0.83160	0.73529

求满足边界条件

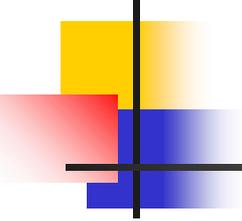
$s'(0) = 0, s'(0.60) = -0.64879$ 三次样条

函数 $s(x)$, 并计算 $s(0.2)$ 。

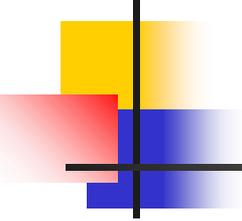
- 
- 解 做差商表(P111),由于是等距离节点,

$$h_i = x_i - x_{i-1} = 0.15 \quad i = 1, 2, 3, 4$$

$$\mu_i = \frac{h_i}{h_{i+1} + h_i} = \frac{1}{2}, \lambda_i = \frac{h_{i+1}}{h_{i+1} + h_i} = \frac{1}{2}$$

- 
- 由第二类边界条件得

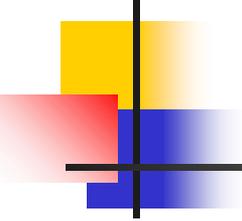
$$\begin{pmatrix} 2 & 1 & & & \\ 0.5 & 2 & 0.5 & & \\ & 0.5 & 2 & 0.5 & \\ & & 0.5 & 2 & 0.5 \\ & & & 1 & 2 \end{pmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \begin{bmatrix} -5.86667 \\ -5.14260 \\ -3.36798 \\ -1.39740 \\ -0.26880 \end{bmatrix}$$

- 
-
- 解方程得

$$M_0 = -2.04462, M_1 = -1.77757, M_2 = -1.13031,$$

$$M_3 = -0.43716, M_4 = 0.08418$$

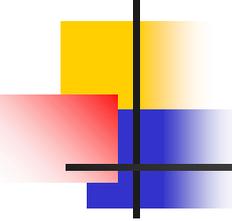
- 将 M_i 代入式(4.4.14)得



$$s(x) = \begin{cases} 0.29672x^3 - 1.02231x^2 + 1, & x \in [0, 0.15] \\ 0.71918x^3 - 1.21242x^2 + 0.02851x + 0.99858, & x \in [0.15, 0.30] \\ 0.77017x^3 - 1.25831x^2 + 0.04228x + 0.99720, & x \in [0.30, 0.45] \\ 0.57927x^3 - 1.00059x^2 + 0.07370x + 1.01461 & x \in [0.45, 0.60] \end{cases}$$

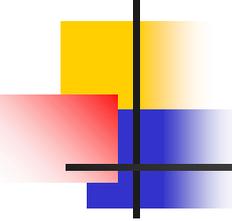
由于 $0.20 \in [0.15, 0.30]$ 故

$$s(0.20) = 0.71918 \times 0.2^3 - 1.21242 \times 0.2^2 + 0.02851 \times 0.2 + 0.99858 = 0.96154$$



4. 5 曲线拟合的最小二乘法

- 插值法是用多项式近似的表示函数,并要求在他们的某些点处的值相拟合.同样也可以用级数的部分和作为函数的近似表达式.无论用那种近似表达式,在实际应用中都要考虑精度,所以我们给出最佳逼近的讨论.



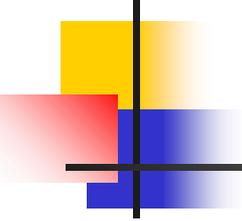
4.5.1 最佳平方逼近

- 定义4.5.1 设 $f(x), g(x) \in C[a, b]$, 称

$$(f, g) \triangleq \int_a^b \rho(x) f(x) g(x) dx$$

为函数 $f(x), g(x)$ 在区间 $[a, b]$ 上的内积.

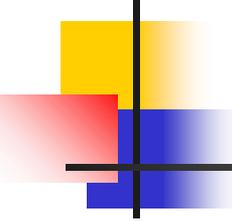
其中 $\rho(x)$ 为区间 $[a, b]$ 上的权函数, 且满足下面两个条件:



(1) 在 $[a, b]$ 上, $\rho(x) \geq 0$, 并且最多只能有有限零点;

(2) $\int_a^b x^i \rho(x) dx$ 存在, $i = 0, 1, 2, \dots$

容易验证, 上述定义的函数内积满足一般内积概念中四条基本性质.



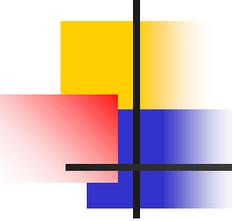
内积的性质

性质1 $(f, g) = (g, f);$

性质2 $(\alpha f, g) = \alpha(f, g), \quad \alpha \in R;$

性质3 $(f_1 + f_2, g) = (f_1, g) + (f_2, g);$

性质4 $(f, f) \geq 0$, 且当且仅当 $f=0$ 是等号成立



函数的欧几里得范数

- 定义4.5.2 设 $f(x), g(x) \in C[a, b]$, 称

$$\|f\|_2 \triangleq \sqrt{(f, f)}$$

为函数 $f(x)$ 的欧几里得范数,或2范数.

以上内容仅为本文档的试下载部分，为可阅读页数的一半内容。如要下载或阅读全文，请访问：<https://d.book118.com/356045211023010122>