Available online at www.sciencedirect.com
ScienceDirect
Journal of Banking & Finance 32 (2008) 1991–2005
Journal of BANKING & FINANCE
www.elsevier.com/locate/jbf

Time-consistency in managing a commodity portfolio: A dynamic risk measure approach

Hélyette Geman [a], Steve Ohana [b,*]

[a] Birkbeck, University of London, ESSEC Business School, Malet Street Bloomsbury, London WC1E 7HX, United Kingdom
[b] Birkbeck, University of London, Malet Street Bloomsbury, London WC1E 7HX, United Kingdom

Received 22 December 2006; accepted 30 May 2007
Available online 15 January 2008

We address the problem of managing a storable commodity portfolio, that includes physical assets and positions in spot and forward markets. The vast amount of capital involved in the acquisition of a power plant or storage facility implies that the financing period stretches over a period of several quarters or years. Hence, an *intertemporally consistent* way of optimizing the portfolio over the planning horizon is required. We demonstrate the temporal inconsistency of static risk objectives based on final wealth and advocate the validity in our setting of a new class of recursive risk measures introduced by Epstein and Zin [Epstein, G., Zin, S., 1989. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. Econometrica, 57 (4) 937–969] and Wang [Wang, T., 2000. A class of dynamic risk measures University of British Columbia]. These risk measures provide important insights on the trade offs between date specific risks (i.e., losses occurring at a point in time) and time duration risks represented by the pair (return, risk) over a planning horizon; in a number of situations, they dramatically improve the efficiency of static risk objectives, as exhibited in numerical examples.

JEL classification: G12; G13

Keywords: Commodity portfolio; Dynamic risk measures; Time consistency; Temporal elasticity of substitution

1. Introduction

We consider the situation of a retailer who is engaged in long-term sale contracts, owns storage facilities and can trade the commodity in illiquid spot and forward markets. The retailer faces a *portfolio optimization problem*, that translates into deciding at each time step which quantity to inject into or withdraw from her storage facilities and trade in the spot and forward market; and a *portfolio valuation problem*, that consists in assessing the value of the global portfolio and each asset composing it. The optimization and the valuation take place in the context of two types of risk: the volume risk that arises from the random demand of long-term customers and is related to exogenous non-traded variables such as weather, and price risk that is linked to the volatility of the commodity price.

The stochastic programming literature, on the one hand, has essentially treated situations where portfolio management is analyzed through a mean–variance criterion applied to final or intermediate wealths, and *fully defined at the first decision date*. In particular, the risk reassessments at intermediate decision dates are not taken into account, leading to possible conflicts across decisions taken over time. Examples of this *open-loop* approach are found in Unger (2002), where a CVaR constraint on the final wealth is addressed through a Monte-Carlo approach, in Martinez-de-Albeniz and Simchi-Levi (2003), where mean–variance trade-offs are considered and yield explicit solutions in a one-period framework, and in Kleindorfer and Li (2005), where the case of a multi-period VaR constraint on cash-flows is examined. The literature on decision theory, on the other hand, has paid a deserved attention to the

* Corresponding author. Tel.: +44 (0) 20 7631 0813; fax: +44 (0) 20 7631 6416.
E-mail address: steve.ohana@gmail.com (S. Ohana).

0378-4266/$ - see front matter
doi:10.1016/j.jbankfin.2007.05.020

problem of dynamic choice under uncertainty. Originally, it was the problem of dynamic consumption planning that was analyzed by economists. In a seminal paper, Epstein and Zin (1989) introduce a set of dynamic utilities, defined recursively in a discrete time setting, and allowing one to separately account for the issue of temporal elasticity of substitution, i.e., controlling consumption over time, and risk aversion, i.e., controlling consumption across random states of nature. In finance, *dynamic risk measures* were recently introduced to account for the occurrence of a stream of random cash-flows over time. A general requirement for these risk measures is their *time-consistency* (see e.g., Artzner et al., 2002; Frittelli and Rosazza Gianin, 2004) since multi-period risks are reevaluated as new information becomes available (see Wang, 2000).

Our article is, to our knowledge, the third attempt after Chen et al. (2004) and Eichhorn and Romisch (2005) to use dynamic risk objectives to manage and evaluate portfolios composed of contracts and physical assets. Eichhorn and Romisch (2005) use a restriction of the set of coherent dynamic risk measures defined by Artzner et al. (2002) to solve an electricity portfolio optimization problem but do not raise the problem of time-consistency of optimal strategies. Chen et al. (2004) define their objective function as an additive intertemporal utility of the consumption process of the portfolio manager. Instead, we choose the more general, Epstein and Zin (1989), non-additive intertemporal utility objective and apply it directly to the cash-flow process. The impact of this change is significant: in our setting, the initial wealth is not a state variable, the only state variables being the inventory level and the positions in the forward market for future delivery periods. In addition, the retailer's problem is posed as a *cash-flow stream management* rather than a consumption planning one. Lastly, the flexibility of the non-additive intertemporal utility allows the portfolio manager to separately control the distribution of cash-flows across time periods and states of nature.[1] The contribution of this paper is to bridge the gap between preference theory, risk management, and stochastic programming in a framework that is intertemporally consistent, computationally feasible, and that provides clear insights on the trade-offs between time-specific risks (e.g., losses that occasion significant internal transactions costs at a point in time) and time-duration risks represented by returns and risks over a planning horizon. More precisely, (i) on the methodological side, we define the concept of time-consistency of optimal strategies, show that the classically used static risk measures depending on net present value are not time-consistent and advocate the use of recursive utilities and (ii) on the operational side, we provide a tractable framework to dynamically manage physical assets under random demand and evolution of spot and forward commodity prices. We show in numerical examples that the use of recursive utilities can help exhibiting a trade-off between final and intermediate wealth risk management. In addition, we demonstrate that strategies based on dynamic risk measures outperform in many situations those based on static risk measures.

The remainder of the paper is organized as follows. In Section 2, we define the time-consistency of optimal strategies and analyze the issues of risk aversion and temporal elasticity of substitution of preferences. In Section 3, we present the portfolio management problem and establish in our setting the fundamental Bellman equation. Section 4 presents a numerical illustration of the main findings. Section 5 contains concluding comments.

2. A comparison of dynamic risk objectives

The objective of this section is to present two examples of dynamic risk preferences and assess their *time-consistency* properties, which we view as an original contribution of the paper.

2.1. Static risk measures

In one period settings, a number of static risk measures have been defined to express preferences of risk averse agents. Mathematically, a (static) risk measure is a function, here denoted v, associating to a contingent claim X a real number $v(X)$. $v(X)$ represents the price that it is acceptable to pay in order to purchase X and $-v(-X)$ represents the capital that must be provisioned in order to make a short position in X acceptable. Static risk measures were first introduced as coherent in the seminal paper by Artzner et al. (1999) to become more generally "convex" in Carr et al. (2001), Frittelli and Rosazza Gianin (2002).

2.2. Risk measure associated to a stream of cash-flows

Defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t))$, the discrete-time stochastic process $G = (G_i)_{i \; 1,\dots,T}$, represents a sequence of random cash-flows occurring at times $(\theta_i)_{i \; 1,\dots,T}$. $\mathscr{G}$ is the set of all $\mathscr{F}_{\theta_i}$-adapted cash-flow processes from $i = 1$ to $i = T$. We choose $\mathscr{F}_{\theta_1} = \{\emptyset, \Omega\}$ as G_1 is deterministic, and $\mathscr{F}_{\theta_T} = \mathscr{F}$, so that full information is revealed at date θ_T.

A dynamic value measure $V = (V_i)_{i \; 1,\dots,T}$ consists of mappings $V_i : \mathscr{G} \times \Omega \to \mathbb{R}$ that associate to each cash-flow process $G \in \mathscr{G}$ and $\omega \in \Omega$ a real number $V_i(G, \omega)$. The resulting stochastic process (V_i) is $\mathscr{F}_{\theta_i}$-adapted. Financially, it represents the value of the sequence of cash-flows $(G_k)_{k \; 1,\dots,T}$ or the capital requirement to cover the liabilities $(-G_k)_{k \; 1,\dots,T}$ at date θ_i.

Let us now propose two categories of dynamic values measures for streams of cash-flows:[2]

[1] Note that the situation of Chen et al. (2004) is obtained as a particular case of our framework, when temporal substitution preferences are ignored.

[2] We restrict our analysis to dynamic value measures $V_i(G)$ depending on present and future cash flows $(G_k)_{k \geqslant i}$ as past cash flows are most of the time considered as sunk costs or secured profits.

1. The first category, found in Riedel (2004) and Weber (2003) and widely used in commodity portfolio management so far (see e.g., Unger, 2002; Kleindorfer and Li, 2005), consists of extensions of static criteria depending on the net present value of the cash-flows between date θ_i and date θ_T:

$$\mathrm{NPV}_{i,T} = \sum_{\tau \geq i}^{T} \beta^{\theta_\tau - \theta_i} G_\tau, \qquad (1)$$
$$V_i(G,\omega) = \mu(\mathrm{NPV}_{i,T} \mid \mathscr{F}_{\theta_i}).$$

   In the above equation, the discount factor β is such that $\beta \leqslant 1$, and μ represents a static risk measure.
2. A second category of criteria (proposed by Epstein and Zin (1989), Wang (2000)) are recursively constructed from the end of the time period by defining:

$$V_T(G,\omega) = G_T,$$
$$V_i(G,\omega) = W(G_i, \mu(V_{i+1} \mid \mathscr{F}_{\theta_i})) \quad \forall\ i \leqslant T-1. \qquad (2)$$

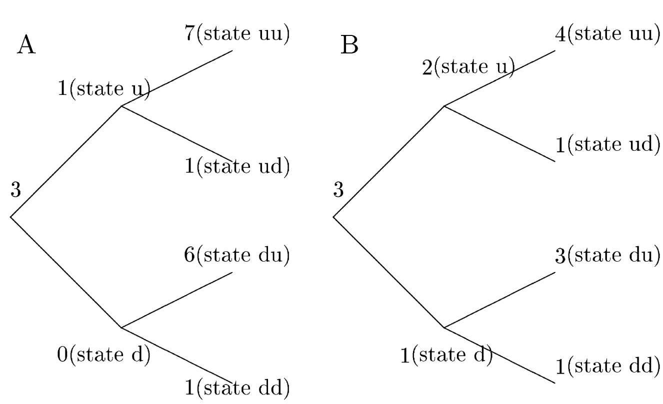
   In the above equation, μ is a one-step *certainty equivalent*.[3] and the mapping $W : \mathbb{R}^2 \to \mathbb{R}$ called an *aggregator*. In this framework, the date θ_i value is assessed recursively by aggregation of the current cash-flow G_i and certainty equivalent of V_{i+1} seen from date θ_i. An important observation to be made at this point is that the process (V_i) is $\mathscr{F}_{\theta_i}$-adapted.

2.3. Time-consistency

Time-consistency is a property which guarantees that preferences implied by a dynamic value measure do not conflict over time.

2.3.1. Examples of time-inconsistency

Consider the two cash-flow streams A and B, where all transition probabilities are supposed to equal 0.5:



[3] We adopt Wang's definition of the certainty equivalent, i.e., a static measure v verifying the monotonicity property (which ensures that if a random variable X is larger than Y almost surely, then $v(X) \geqslant v(Y)$), and reduced to the identity on the space of constant random variables.

Let us evaluate stream A using the dynamic value measure (1) with $\mu(X) = u^{-1}(\mathbb{E}[u(X)])$, $u(x) = \ln(x)$, and $\beta = 1$:

$$V_2(A,u) = \exp(\mathbb{E}(\ln(\mathrm{NPV}^A_{2,3}) \mid u))$$
$$= \exp(0.5(\ln(8) + \ln(2))) = 4;$$
$$V_2(A,d) = \exp(\mathbb{E}(\ln(\mathrm{NPV}^A_{2,3}) \mid d)) = \sqrt{6}$$
$$V_1(A) = \exp(\mathbb{E}(\ln(\mathrm{NPV}^A_{1,3})))$$
$$= \exp(0.25(\ln(11) + \ln(5) + \ln(9) + \ln(4)))$$
$$= (55 \times 36)^{\frac{1}{4}}.$$

Now evaluate stream B:

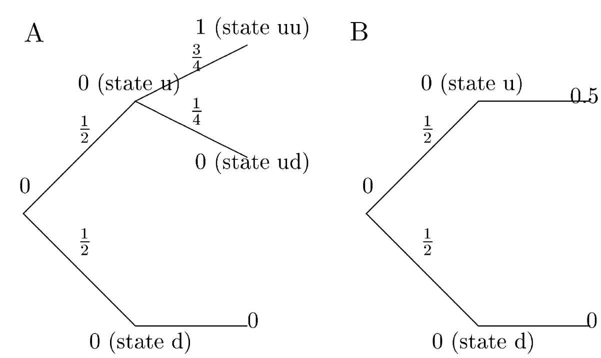
$$V_2(B,u) = \exp(\mathbb{E}(\ln(\mathrm{NPV}^B_{2,3}) \mid u))$$
$$= \exp(0.5(\ln(6) + \ln(3))) = \sqrt{18};$$
$$V_2(B,d) = \exp(\mathbb{E}(\ln(\mathrm{NPV}^B_{2,3}) \mid d)) = \sqrt{8}$$
$$V_1(B) = \exp(\mathbb{E}(\ln(\mathrm{NPV}^B_{1,3})))$$
$$= \exp(0.25(\ln(9) + \ln(6) + \ln(7) + \ln(5)))$$
$$= (54 \times 35)^{\frac{1}{4}}.$$

We thus have simultaneously the following inequalities:

$$V_2(A,u) < V_2(B,u) \quad \text{and} \quad V_2(A,d) < V_2(B,d)$$
$$\text{while} \quad V_1(A) > V_1(B).$$

As in addition $A_1 = B_1 = 3$, the value measure V is not time consistent.

Time-consistency does not hold either if μ is a mean variance instead of an expected utility criterion in Eq. (1). To see this, consider the two following cash-flow streams A (left) and B (right), with transition probabilities displayed on top of each arc:



Let us evaluate stream A using the dynamic value measure (1) with $\mu(X) = \mathbb{E}(X) - \mathrm{Var}(X)$:

$$V_2(A,u) = \mathbb{E}\left(\mathrm{NPV}^A_{2,3} \mid u\right) - \mathrm{Var}\left(\mathrm{NPV}^A_{2,3} \mid u\right)$$
$$= \frac{3}{4} - \left(\frac{3}{4} - \frac{9}{16}\right) = \frac{9}{16},$$
$$V_2(A,d) = \mathbb{E}\left(\mathrm{NPV}^A_{2,3} \mid d\right) - \mathrm{Var}\left(\mathrm{NPV}^A_{2,3} \mid d\right) = 0,$$

$$V_1(A) = \mathbb{E}\Big(\mathrm{NPV}^A_{1,3}\Big) - \mathrm{Var}\Big(\mathrm{NPV}^A_{1,3}\Big) = \frac{1}{2}\times\frac{3}{4} - \left(\frac{3}{8}-\frac{9}{64}\right) = \frac{9}{64}.$$

Now evaluate stream B:

$$V_2(B,u) = \mathbb{E}\Big(\mathrm{NPV}^B_{2,3} \mid u\Big) - \mathrm{Var}\Big(\mathrm{NPV}^B_{2,3} \mid u\Big) = \frac{1}{2},$$

$$V_2(B,d) = \mathbb{E}\Big(\mathrm{NPV}^B_{2,3} \mid d\Big) - \mathrm{Var}\Big(\mathrm{NPV}^B_{2,3} \mid d\Big) = 0,$$

$$V_1(B) = \mathbb{E}\Big(\mathrm{NPV}^B_{1,3}\Big) - \mathrm{Var}\Big(\mathrm{NPV}^B_{1,3}\Big) = \frac{1}{2}\times\frac{1}{2} - \left(\frac{1}{2}\times\frac{1}{4}-\frac{1}{16}\right) = \frac{3}{16} = \frac{12}{64}.$$

We thus have the simultaneous inequalities:

$$V_2(A,u) > V_2(B,u) \quad \text{and} \quad V_2(A,d) \geqslant V_2(B,d) \quad \text{while} \quad V_1(A) < V_1(B).$$

Let us denote $\mathscr{H}_t(\omega)$ the set of events $\omega' \in \Omega$ having the same history as ω up to time t[4] and formally define the time-consistency property:

Definition 2.1. The dynamic value measure V is intrinsically time consistent if
for all $A, B \in \mathscr{G}, t \in \mathscr{T} = \{1,\ldots,T\}, \omega \in \Omega$,

$$\begin{cases} A(t,\omega) \geqslant B(t,\omega) \\ \forall \omega' \in \mathscr{H}_t(\omega),\ V_{t+1}(A,\omega') \geqslant V_{t+1}(B,\omega') \end{cases}$$
$$\Rightarrow\ V_t(A,\omega) \geqslant V_t(B,\omega),$$

Property 2.2. *If the aggregator W is monotonic, then the dynamic value measures of the recursive type (2) are intrinsically time-consistent.*

Proof. For any $t \in \mathscr{T}, \omega \in \Omega$, if $A(t,\omega) \geqslant B(t,\omega)$ and $V_{t+1}(A,\omega') \geqslant V_{t+1}(B,\omega')\ \forall\ \omega' \in \mathscr{H}_t(\omega)$, then, by monotonicity of certainty equivalents, $\mu(V_{t+1}(A,.) \mid \mathscr{F}_t)(\omega) \geqslant \mu(V_{t+1}(B,.) \mid \mathscr{F}_t)(\omega)$. In turn, by monotonicity of the aggregator W,

$$V_t(A,\omega) = W(A(t,\omega), \quad \mu(V_{t+1}(A,.) \mid \mathscr{F}_t)(\omega)) \geqslant W(B(t,\omega), \quad \mu(V_{t+1}(B,.) \mid \mathscr{F}_t)(\omega)) = V_t(B,\omega). \quad \square$$

2.3.2. Time-consistency of optimal strategies and comparison of criteria

In the previous section, we defined an *intrinsic* time-consistency property, related to the evaluation of *exogenous* streams of random cash-flows. In this section, we assume instead that *the cash-flows depend on decisions that are made at each date* θ_i, using the information available at this date. A decision at date θ_i is the result of the optimization of a dynamic value measure of the type described above. This optimization not only yields the first decision at that date, but a whole *decision planning* for all subsequent stages. The question we pose in this section is the following: are these optimal plannings consistent over time?

Let us define the problem formally: consider a cash-flow sequence $(G_i)_{1\leqslant i\leqslant T}$, occurring at dates $(\theta_i)_{i\geqslant 1}$, depending on decisions $(q_i)_{1\leqslant i\leqslant T}$ and a multi-dimensional random process $(\xi_i)_{1\leqslant i\leqslant T} : G_i := f(q_i, \xi_i)$. The process (ξ_i) is assumed to be of the type $\xi_{i+1} = g(\xi_i, \epsilon_{i+1})$ for some reasonably behaved function g, and an $(\mathscr{F}_{\theta_i})$-adapted white noise process (ϵ_i).

We introduce the state variables x_i on which depend decisions at time θ_i and denote $\mathscr{A}(x_i)$ the set of admissible strategies $(q_k)_{i\leqslant k\leqslant T}$ at time θ_i. We suppose that, after decision q_i is made at time θ_i, the state x_i leads to $x_{i+1} = h(x_i, q_i, \epsilon_{i+1})$, where h is a deterministic function; (q_i) is supposed to be an $(\mathscr{F}_{\theta_i})$-adapted process.

Lastly, we consider the following optimization problem, related to a dynamic value measure V:

$$J_i(x_i) = \underset{(q_k)_{k\geqslant i}\in\mathscr{A}(x_i)}{\mathrm{Max}} V_i(G). \tag{3}$$

We denote $(q_k^{*i}(x_i))_{k\geqslant i}$ the resulting $(\mathscr{F}_{\theta_k})$-adapted optimal strategy decided at date θ_i.[5] The question of consistency of optimal strategies can be formulated in the following way:

Is $q^{*i}_{i+1}(x_i, \epsilon_{i+1})$ equal to $q^{*(i+1)}_{i+1}(x_{i+1})$,
where $x_{i+1} = h(x_i, q_i^{*i}(x_i), \epsilon_{i+1})$?

We now turn to the time-consistency of optimal strategies derived from the two dynamic value measures defined above.

First, let us consider the final wealth objective defined in Eq. (1) with $\mu(X) = u^{-1}(\mathbb{E}[u(X)])$, i.e, $V_i(G,\omega) = u^{-1}(\mathbb{E}(u(\mathrm{NPV}_{i,T}) \mid \mathscr{F}_{\theta_i}))$:[6]

$$J_i(x_i) = \underset{(q_k)_{k\geqslant i}\in\mathscr{A}(x_i)}{\mathrm{Max}} V_i(G)$$
$$= u^{-1}\left(\underset{q_i}{\mathrm{Max}}\ \underset{(q_k)_{k\geqslant i+1}}{\mathrm{Max}}\ \mathbb{E}_{\theta_i}(\mathbb{E}_{\theta_{i+1}}(u(\mathrm{NPV}_{i,T})))\right)$$
$$= u^{-1}\left(\underset{q_i}{\mathrm{Max}}\, \mathbb{E}_{\theta_i}\left(\underset{(q_k)_{k\geqslant i+1}\in\mathscr{A}(x_{i+1})}{\mathrm{Max}} \mathbb{E}_{\theta_{i+1}}(u(\mathrm{NPV}_{i,T}))\right)\right).$$

The date θ_{i+1} implied problem $\underset{(q_k)_{k\geqslant i+1}}{\mathrm{Max}}\ \mathbb{E}_{\theta_{i+1}}(u(\mathrm{NPV}_{i,T}))$ differs from the one derived from the dynamic value measure (V_i), i.e., $\underset{(q_k)_{k\geqslant i+1}}{\mathrm{Max}} V_{i+1}(G) = u^{-1}\left(\underset{(q_k)_{k\geqslant i+1}}{\mathrm{Max}}\ \mathbb{E}_{\theta_{i+1}}(u(\mathrm{NPV}_{i+1,T}))\right)$.[7] As a result, the optimal strategy decided at time θ_i differs from the optimal strategy exhibited at time θ_{i+1}.

[4] Intuitively, it represents the set of all possible subsequent events after time t branching from a given scenario ω.

[5] We classically suppose throughout this section that all encountered optimization problems have a unique solution.

[6] From now on, we denote $\mathbb{E}(X \mid \mathscr{F}_{\theta_i})$ $\mathbb{E}_{\theta_i}(X)$.

[7] The particular cases of a linear utility $u(x)$ x (with arbitrary $\beta \leqslant 1$) or CARA utility $u(x)$ $e^{-\lambda x}$ (with β 1) yield time consistent optimal strategies since in both cases: $V_t(G)$ $G_t + \beta^{\theta_{i+1}-\theta_i}\mu[V_{t+1} \mid \mathscr{F}_t]$ with $\mu(X)$ $u^{-1}(\mathbb{E}[u(X)])$; therefore, a Bellman equation linking optimal strategies at times θ_i and θ_{i+1} is derived, as will be shown in the end of this section.

In order to investigate the issue of whether time-inconsistency remains if we use a mean variance objective instead of an expected utility, we consider a sequence of three cash-flows (G_1, G_2, G_3), depending on the process $(\xi_{\theta_i})_{i=1,2,3}$ and $\mathscr{F}_{\theta_i}$-measurable decisions $(q_i)_{i=1,2,3}$, and decompose the variance of the sum of these cash-flows (here, we suppose $\beta = 1$ for simplicity):

$$\begin{aligned}
&\mathrm{Var}_{\theta_1}(G_1+G_2+G_3)\\
&\quad= \mathrm{Var}_{\theta_1}(G_2+G_3) = \mathbb{E}_{\theta_1}[(G_2+G_3)^2] - [\mathbb{E}_{\theta_1}(G_2+G_3)]^2\\
&\quad= \mathbb{E}_{\theta_1}[\mathbb{E}_{\theta_2}((G_2+G_3)^2)] - [\mathbb{E}_{\theta_1}(\mathbb{E}_{\theta_2}(G_2+G_3))]^2\\
&\quad= \mathbb{E}_{\theta_1}[\mathbb{E}_{\theta_2}((G_2+G_3)^2)] - \mathbb{E}_{\theta_1}([\mathbb{E}_{\theta_2}(G_2+G_3)]^2)\\
&\qquad+ \mathbb{E}_{\theta_1}([\mathbb{E}_{\theta_2}(G_2+G_3)]^2) - [\mathbb{E}_{\theta_1}(\mathbb{E}_{\theta_2}(G_2+G_3))]^2\\
&\quad= \mathbb{E}_{\theta_1}[\mathrm{Var}_{\theta_2}(G_2+G_3)] + \mathrm{Var}_{\theta_1}(\mathbb{E}_{\theta_2}(G_2+G_3))\\
&\quad= \mathbb{E}_{\theta_1}[\mathrm{Var}_{\theta_2}(G_3)] + \mathrm{Var}_{\theta_1}(G_2+\mathbb{E}_{\theta_2}(G_3)).
\end{aligned}$$

The last equality illuminates why total variance is time inconsistent: the $\mathscr{F}_{\theta_1}$-measurable term $\mathrm{Var}_{\theta_1}(G_2+\mathbb{E}_{\theta_2}(G_3))$ is impacted by decisions q_1, q_2, and q_3, in contrast to the term G_1, which depends only on decision q_1. This fact compromises the existence of any dynamic programming equation of the Bellman type linking optimal strategies at dates θ_1 and θ_2:

$$\begin{aligned}
J_1(x_1) &= \underset{(q_k)_{k=1,2,3}\in\mathscr{A}(x_1)}{\mathrm{Max}} \left\{\mathbb{E}_{\theta_1}(G_1+G_2+G_3) - \mathrm{Var}_{\theta_1}(G_1+G_2+G_3)\right\}\\
&= \underset{(q_k)_{k=1,2,3}}{\mathrm{Max}} \left\{G_1(q_1) - \mathrm{Var}_{\theta_1}(G_2+\mathbb{E}_{\theta_2}(G_3))\right.\\
&\qquad \left.+\mathbb{E}_{\theta_1}(\mathbb{E}_{\theta_2}(G_2+G_3) - \mathrm{Var}_{\theta_2}(G_3))\right\}\\
&\neq \underset{q_1}{\mathrm{Max}}\left\{G_1(q_1) - \mathrm{Var}_{\theta_1}(G_2+\mathbb{E}_{\theta_2}(G_3))\right.\\
&\qquad \left.+\mathbb{E}_{\theta_1}\left(\underset{(q_k)_{k=2,3}\in\mathscr{A}(x_2)}{\mathrm{Max}} \mathbb{E}_{\theta_2}(G_2+G_3) - \mathrm{Var}_{\theta_2}(G_3)\right)\right\}.
\end{aligned}$$

We now turn to the dynamic value measures described in Eq. (2).

As a first observation, let us consider the case of a linear aggregator $W(x,y) = x+y$. The date θ_i objective derived from the value measure V_i defined in Eq. (2) is then:

$$\begin{aligned}
J_i(x_i) &= \underset{(q_k)_{k\geqslant i}\in\mathscr{A}(x_i)}{\mathrm{Max}} V_i(G) = \underset{(q_k)_{k\geqslant i}}{\mathrm{Max}}\left\{G_i(q_i) + \mu_{\theta_i}(V_{i+1})\right\}\\
&= \underset{q_i}{\mathrm{Max}}\left\{G_i(q_i) + \underset{(q_k)_{k\geqslant i+1}\in\mathscr{A}(x_{i+1})}{\mathrm{Max}} \mu_{\theta_i}(V_{i+1})\right\}.
\end{aligned}$$

The question at this stage is to know whether the following property holds:

$$\underset{(q_k)_{k\geqslant i+1}}{\mathrm{Max}} \mu_{\theta_i}(V_{i+1}) \overset{?}{=} \mu_{\theta_i}\left(\underset{(q_k)_{k\geqslant i+1}}{\mathrm{Max}} V_{i+1}\right). \tag{4}$$

If the permutation is valid, then the optimal strategies will be time-consistent since the date θ_{i+1} implied problem $\underset{(q_k)_{k\geqslant i+1}}{\mathrm{Max}} V_{i+1}$ will coincide with the optimization problem at stage $i+1$; otherwise, they will not.

Let us introduce the following aggregator W and certainty equivalent μ:

$$\begin{cases} W(x,y) = \phi^{-1}(\phi(x) + \beta\phi(y)),\\ \mu(X) = u^{-1}(\mathbb{E}[u(X)]). \end{cases} \tag{5}$$

where u and ϕ are increasing functions and β a positive discounting factor:[8]

$$\begin{aligned}
J_i(x_i) &= \underset{(q_k)_{k\geqslant i}\in\mathscr{A}(x_i)}{\mathrm{Max}} V_i(G)\\
&= \underset{(q_k)_{k\geqslant i}\in\mathscr{A}(x_i)}{\mathrm{Max}} \phi^{-1}(\phi(G_i(q_i)) + \beta\phi(\mu_{\theta_i}(V_{i+1})))\\
&= \phi^{-1}\left(\underset{(q_k)_{k\geqslant i}\in\mathscr{A}(x_i)}{\mathrm{Max}} \left\{\phi(G_i(q_i)) + \beta\phi(\mu_{\theta_i}(V_{i+1}))\right\}\right)\\
&= \phi^{-1}\left(\underset{q_i}{\mathrm{Max}}\left\{\phi(G_i(q_i)) + \beta\phi(\underset{(q_k)_{k\geqslant i+1}}{\mathrm{Max}} \mu_{\theta_i}(V_{i+1}))\right\}\right).
\end{aligned}$$

The inversion between operators Max and μ in the last equality is permitted as

$$\begin{aligned}
\underset{(q_k)_{k\geqslant i+1}}{\mathrm{Max}} \mu_{\theta_i}(V_{i+1}) &= \underset{(q_k)_{k\geqslant i+1}}{\mathrm{Max}} u^{-1}(\mathbb{E}_{\theta_i}(u(V_{i+1})))\\
&= u^{-1}\left(\mathbb{E}_{\theta_i}\left(\underset{(q_k)_{k\geqslant i+1}\in\mathscr{A}(x_{i+1})}{\mathrm{Max}} u(V_{i+1})\right)\right)\\
&= u^{-1}\left(\mathbb{E}_{\theta_i}\left(u\left(\underset{(q_k)_{k\geqslant i+1}\in\mathscr{A}(x_{i+1})}{\mathrm{Max}} V_{i+1}\right)\right)\right)\\
&= \mu_{\theta_i}\left(\underset{(q_k)_{k\geqslant i+1}\in\mathscr{A}(x_{i+1})}{\mathrm{Max}} V_{i+1}\right).
\end{aligned}$$

We can now present a general sufficient condition of time-consistency for optimal strategies:

Property 2.3. *If there exist non-decreasing functions $(a_i)(b_i), (c_i), (d_i)$ and positive numbers (β_i) such that*

$$V_i(G) = a_i\langle b_i(G_i(q_i)) + \beta_i c_i[\mathbb{E}_{\theta_i}(d_i(V_{i+1}(G)))]\rangle, \tag{6}$$

then the dynamic value measure (V_i) leads to time-consistent optimal strategies.

For the recursive utilities defined in (5), Eq. (6) holds with for all i, $a_i = \phi^{-1}, b_i = \phi, c_i = \phi\circ u^{-1}, d_i = u$, and $\beta_i = \beta$. In the classical case of expectation maximization (risk-neutrality), Eq. (6) holds with $a_i = b_i = c_i = d_i = Id$.

2.4. Risk aversion and temporal elasticity of substitution

We have mentioned earlier that the problem of dynamic valuation under uncertainty involves two dimensions, one with respect to the distribution of cash-flows across states of nature, the other over consecutive time periods. The first dimension has an effect on *the final wealth distribution* while the second one impacts *the likelihood of bankruptcy or other high transaction costs within the time period.*

Dynamic value measures defined in Eq. (1) only depend on the NPV of future cash-flows, hence assume infinite temporal elasticity of substitution between cash-flows

[8] This particular choice for the aggregator and the certainty equivalent was first suggested by Epstein and Zin (1989) and later on extended by Wang (2000) to incorporate ambiguity aversion.