

第四章

例题及习题



例1. 设曲线经过点(1, 2), 且其上任一点处切线斜率等于该点横坐标两倍, 求此曲线方程.

解:

$$0 \quad y' = 2x$$

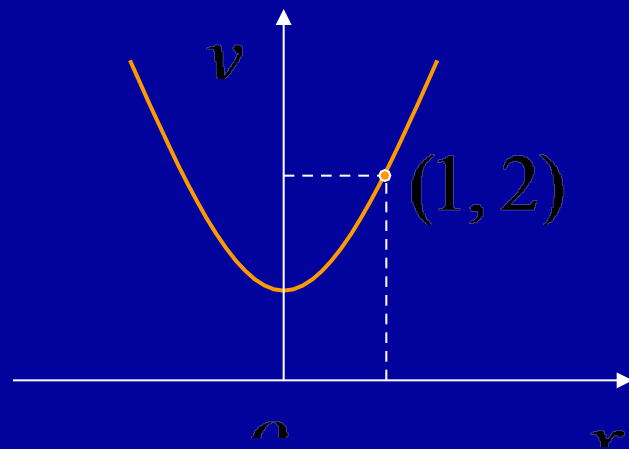
$$\therefore y = \int 2x dx = x^2 + C$$

所求曲线过点(1, 2), 故有

$$2 = 1^2 + C$$

$$\therefore C = 1$$

所以所求曲线为 $y = x^2 + 1$



例2. 求 $\int \frac{dx}{x\sqrt[3]{x}}$.

解: 原式 $= \int x^{-\frac{4}{3}} dx = \frac{x^{-\frac{4}{3}+1}}{-\frac{4}{3}+1} + C$
 $= -3x^{-\frac{1}{3}} + C$

例3. 求 $\int \sin \frac{x}{2} \cos \frac{x}{2} dx$.

解: 原式 $= \int \frac{1}{2} \sin x dx = -\frac{1}{2} \cos x + C$



例4. 求 $\int 2^x(e^x - 5)dx$.

解: 原式 = $\int [(2e)^x - 5 \cdot 2^x] dx$

$$= \frac{(2e)^x}{\ln(2e)} - 5 \frac{2^x}{\ln 2} + C$$

$$= 2^x \left[\frac{e^x}{\ln 2 + 1} - \frac{5}{\ln 2} \right] + C$$



例5. 求 $\int \tan^2 x dx$.

解: 原式 = $\int (\sec^2 x - 1) dx$
 $= \int \sec^2 x dx - \int dx = \tan x - x + C$

例6. 求 $\int \frac{1+x+x^2}{x(1+x^2)} dx$.

解: 原式 = $\int \frac{x+(1+x^2)}{x(1+x^2)} dx$
 $= \int \frac{1}{1+x^2} dx + \int \frac{1}{x} dx$
 $= \arctan x + \ln|x| + C$



例7. 求 $\int \frac{x^4}{1+x^2} dx$.

解: 原式 = $\int \frac{(x^4 - 1) + 1}{1 + x^2} dx$

= $\int \frac{(x^2 - 1)(x^2 + 1) + 1}{1 + x^2} dx$

= $\int (x^2 - 1) dx + \int \frac{dx}{1 + x^2}$

= $\frac{1}{3}x^3 - x + \arctan x + C$



思索与练习

1. 证实 $\frac{1}{2}e^{2x}$, $e^x \operatorname{sh} x$, $e^x \operatorname{ch} x$ 都是 $\frac{e^x}{\operatorname{ch} x - \operatorname{sh} x}$ 的原函数

提醒: $\operatorname{ch} x = \frac{e^x + e^{-x}}{2}$, $\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$

2. 若 e^{-x} 是 $f(x)$ 的原函数则

$$\int x^2 f(\ln x) dx = \underline{\underline{-\frac{1}{2}x^2 + C}}$$

提醒: $f(x) = (e^{-x})' = -e^{-x}$

$$f(\ln x) = -e^{-\ln x} = -\frac{1}{x}$$



3. 若 $f(x)$ 导函数为 $\sin x$, 则 $f(x)$ 一个原函数是 (**B**).

(A) $1 + \sin x$; (B) $1 - \sin x$;

(C) $1 + \cos x$; (D) $1 - \cos x$.

提醒: 已知 $f'(x) = \sin x$
求 $(?)' = f(x)$
即 $(?)'' = \sin x$



4. 求以下积分:

$$(1) \int \frac{dx}{x^2(1+x^2)};$$

$$(2) \int \frac{dx}{\sin^2 x \cos^2 x}.$$

提醒:

$$(1) \frac{1}{x^2(1+x^2)} = \frac{(1+x^2) - x^2}{x^2(1+x^2)} = \frac{1}{x^2} - \frac{1}{1+x^2}$$

$$(2) \frac{1}{\sin^2 x \cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \\ = \sec^2 x + \csc^2 x$$



5. 求不定积分 $\int \frac{e^{3x} + 1}{e^x + 1} dx$.

解:
$$\int \frac{e^{3x} + 1}{e^x + 1} dx$$
$$= \int \frac{(e^x + 1)(e^{2x} - e^x + 1)}{e^x + 1} dx$$
$$= \int (e^{2x} - e^x + 1) dx$$
$$= \frac{1}{2}e^{2x} - e^x + x + C$$



例1. 求 $\int (ax+b)^m dx$ ($m \neq -1$).

解: 令 $u = ax+b$, 则 $du = a dx$, 故

$$\begin{aligned} \text{原式} &= \int u^m \frac{1}{a} du = \frac{1}{a} \cdot \frac{1}{m+1} u^{m+1} + C \\ &= \frac{1}{a(m+1)} (ax+b)^{m+1} + C \end{aligned}$$

注: 当 $m = -1$ 时

$$\int \frac{dx}{ax+b} = \frac{1}{a} \int \frac{d(ax+b)}{ax+b} = \frac{1}{a} \ln|ax+b| + C$$



例2. 求 $\int \frac{dx}{a^2 + x^2}$.

解: $\int \frac{dx}{a^2 + x^2} = \frac{1}{a^2} \int \frac{dx}{1 + (\frac{x}{a})^2}$

↓
令 $u = \frac{x}{a}$, 则 $du = \frac{1}{a} dx$

$$\frac{1}{a} \int \frac{du}{1+u^2} = \frac{1}{a} \arctan u + C$$

$$= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

想到公式

$$\int \frac{du}{1+u^2} = \arctan u + C$$



例3. 求 $\int \frac{dx}{\sqrt{a^2 - x^2}}$ ($a > 0$).

解:
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{dx}{a\sqrt{1 - (\frac{x}{a})^2}} = \int \frac{d(\frac{x}{a})}{\sqrt{1 - (\frac{x}{a})^2}}$$
$$= \arcsin \frac{x}{a} + C$$

想到
$$\int \frac{du}{\sqrt{1 - u^2}} = \arcsin u + C$$

$$\int f[\varphi(x)]\varphi'(x)dx = \int f(\varphi(x))d\varphi(x) \quad (\text{直接凑微分})$$



例4. 求 $\int \tan x dx$.

解:
$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{d\cos x}{\cos x}$$
$$= -\ln|\cos x| + C$$

类似

$$\int \cot x dx = \int \frac{\cos x dx}{\sin x} = \int \frac{d\sin x}{\sin x}$$
$$= \ln|\sin x| + C$$



例5. 求 $\int \frac{dx}{x^2 - a^2}$.

解:

$$Q \frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a} \left(\frac{1}{x-a} - \frac{1}{x+a} \right)$$

$$\therefore \text{原式} = \frac{1}{2a} \left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a} \right]$$

$$= \frac{1}{2a} \left[\int \frac{d(x-a)}{x-a} - \int \frac{d(x+a)}{x+a} \right]$$

$$= \frac{1}{2a} \left[\ln|x-a| - \ln|x+a| \right] + C = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$



惯用几个配元形式:

$$(1) \int f(ax+b)dx = \frac{1}{a} \int f(ax+b) d(ax+b)$$

$$(2) \int f(x^n)x^{n-1} dx = \frac{1}{n} \int f(x^n) dx^n$$

$$(3) \int f(x^n)\frac{1}{x} dx = \frac{1}{n} \int f(x^n) \frac{1}{x^n} dx^n$$

万能凑幂法

$$(4) \int f(\sin x)\cos x dx = \int f(\sin x) d\sin x$$

$$(5) \int f(\cos x)\sin x dx = \int f(\cos x) d\cos x$$



$$(6) \int f(\tan x) \sec^2 x dx = \int f(\tan x) d \tan x$$

$$(7) \int f(e^x) e^x dx = \int f(e^x) de^x$$

$$(8) \int f(\ln x) \frac{1}{x} dx = \int f(\ln x) d \ln x$$

例6. 求 $\int \frac{dx}{x(1+2 \ln x)}$.

解: 原式 = $\int \frac{d \ln x}{1+2 \ln x} = \frac{1}{2} \int \frac{d(1+2 \ln x)}{1+2 \ln x}$

$$= \frac{1}{2} \ln |1+2 \ln x| + C$$



例7. 求 $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}} dx$.

解: 原式 $= 2 \int e^{3\sqrt{x}} d\sqrt{x} = \frac{2}{3} \int e^{3\sqrt{x}} d(3\sqrt{x})$
 $= \frac{2}{3} e^{3\sqrt{x}} + C$

例8. 求 $\int \sec^6 x dx$.

解: 原式 $= \int (\tan^2 x + 1)^2 d \tan x$
 $= \int (\tan^4 x + 2 \tan^2 x + 1) d \tan x$
 $= \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x + C$



例9. 求 $\int \frac{dx}{1+e^x}$.

解法1

$$\begin{aligned}\int \frac{dx}{1+e^x} &= \int \frac{(1+e^x) - e^x}{1+e^x} dx = \int dx - \int \frac{d(1+e^x)}{1+e^x} \\ &= x - \ln(1+e^x) + C\end{aligned}$$

解法2

$$\begin{aligned}\int \frac{dx}{1+e^x} &= \int \frac{e^{-x}}{1+e^{-x}} dx = -\int \frac{d(1+e^{-x})}{1+e^{-x}} \\ &= -\ln(1+e^{-x}) + C\end{aligned}$$

$$-\ln(1+e^{-x}) = -\ln[e^{-x}(e^x+1)] \quad \text{两法结果一样}$$



例10. 求 $\int \sec x dx$.

解法1

$$\begin{aligned}\int \sec x dx &= \int \frac{\cos x}{\cos^2 x} dx = \int \frac{d \sin x}{1 - \sin^2 x} \\ &= \frac{1}{2} \int \left[\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} \right] d \sin x \\ &= \frac{1}{2} \left[\ln |1 + \sin x| - \ln |1 - \sin x| \right] + C \\ &= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C\end{aligned}$$



$$\begin{aligned}
 \text{解法 2} \quad \int \sec x dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx \\
 &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\
 &= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} \\
 &= \ln |\sec x + \tan x| + C
 \end{aligned}$$

一样可证

$$\int \csc x dx = \ln |\csc x - \cot x| + C$$

或

$$\int \csc x dx = \ln \left| \tan \frac{x}{2} \right| + C$$



例11. 求 $\int \frac{x^3}{(x^2 + a^2)^{3/2}} dx$.

解: 原式 $= \frac{1}{2} \int \frac{x^2 dx^2}{(x^2 + a^2)^{3/2}} = \frac{1}{2} \int \frac{(x^2 + a^2) - a^2}{(x^2 + a^2)^{3/2}} dx^2$

$$= \frac{1}{2} \int (x^2 + a^2)^{-1/2} d(x^2 + a^2)$$
$$- \frac{a^2}{2} \int (x^2 + a^2)^{-3/2} d(x^2 + a^2)$$
$$= \sqrt{x^2 + a^2} + \frac{a^2}{\sqrt{x^2 + a^2}} + C$$



例12. 求 $\int \cos^4 x dx$.

解:
$$\begin{aligned} Q \cos^4 x &= (\cos^2 x)^2 = \left(\frac{1 + \cos 2x}{2}\right)^2 \\ &= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x) \\ &= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1 + \cos 4x}{2}\right) \\ &= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) \end{aligned}$$

$$\begin{aligned} \therefore \int \cos^4 x dx &= \frac{1}{4} \int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx \\ &= \frac{1}{4} \left[\frac{3}{2} \int dx + \int \cos 2x d(2x) + \frac{1}{8} \int \cos 4x d(4x) \right] \\ &= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C \end{aligned}$$



例13. 求 $\int \sin^2 x \cos^2 3x dx$.

解: $\circ \sin^2 x \cos^2 3x = [\frac{1}{2}(\sin 4x - \sin 2x)]^2$

$$= \frac{1}{4} \sin^2 4x - \frac{1}{4} \cdot 2 \sin 4x \sin 2x + \frac{1}{4} \sin^2 2x$$
$$= \frac{1}{8}(1 - \cos 8x) - \sin^2 2x \cos 2x + \frac{1}{8}(1 - \cos 4x)$$

$$\therefore \text{原式} = \frac{1}{4} \int dx - \frac{1}{64} \int \cos 8x d(8x)$$
$$- \frac{1}{2} \int \sin^2 2x d(\sin 2x) - \frac{1}{32} \int \cos 4x d(4x)$$
$$= \frac{1}{4} x - \frac{1}{64} \sin 8x - \frac{1}{6} \sin^3 2x - \frac{1}{32} \sin 4x + C$$



例14. 求 $\int \frac{x+1}{x(1+xe^x)} dx$.

解: 原式 = $\int \frac{(x+1)e^x}{xe^x(1+xe^x)} dx = \int \left(\frac{1}{xe^x} - \frac{1}{1+xe^x} \right) d(xe^x)$

$$= \ln |xe^x| - \ln |1+xe^x| + C$$
$$= x + \ln |x| - \ln |1+xe^x| + C$$

分析: $\frac{1}{xe^x(1+xe^x)} = \frac{1+xe^x - xe^x}{xe^x(1+xe^x)} = \frac{1}{xe^x} - \frac{1}{1+xe^x}$

$$(x+1)e^x dx = xe^x dx + e^x dx = d(xe^x)$$



例15. 求 $\int \left[\frac{f(x)}{f'(x)} - \frac{f''(x)f^2(x)}{f'^3(x)} \right] dx$.

解: 原式 $= \int \frac{f(x)}{f'(x)} \left[1 - \frac{f''(x)f(x)}{f'^2(x)} \right] dx$

$$= \int \frac{f(x)}{f'(x)} \cdot \frac{f'^2(x) - f''(x)f(x)}{f'^2(x)} dx$$
$$= \int \frac{f(x)}{f'(x)} d\left(\frac{f(x)}{f'(x)}\right)$$
$$= \frac{1}{2} \left[\frac{f(x)}{f'(x)} \right]^2 + C$$



小结 惯用简化技巧:

(1) 分项积分: 利用积化和差; 分式分项;

$$1 = \sin^2 x + \cos^2 x \text{ 等恒等式}$$

(2) 降低幂次: 利用倍角公式, 如

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x); \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x);$$

万能凑幂法

$$\left\{ \begin{array}{l} \int f(x^n) x^{n-1} dx = \frac{1}{n} \int f(x^n) d x^n \\ \int f(x^n) \frac{1}{x} dx = \frac{1}{n} \int f(x^n) \frac{1}{x^n} d x^n \end{array} \right.$$

(3) 统一函数: 利用三角公式; 配元方法

(4) 巧妙换元或配元



思索与练习

1. 以下各题求积方法有何不一样?

$$(1) \int \frac{dx}{4+x} = \int \frac{d(4+x)}{4+x}$$

$$(2) \int \frac{dx}{4+x^2} = \frac{1}{2} \int \frac{d(\frac{x}{2})}{1+(\frac{x}{2})^2}$$

$$(3) \int \frac{x}{4+x^2} dx = \frac{1}{2} \int \frac{d(4+x^2)}{4+x^2}$$

$$(4) \int \frac{x^2}{4+x^2} dx = \int \left[1 - \frac{4}{4+x^2} \right] dx$$

$$(5) \int \frac{dx}{4-x^2} = \frac{1}{4} \int \left[\frac{1}{2-x} + \frac{1}{2+x} \right] dx$$

$$(6) \int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{d(x-2)}{\sqrt{4-(x-2)^2}}$$



2. 求 $\int \frac{dx}{x(x^{10}+1)}$.

提醒:

法1
$$\int \frac{dx}{x(x^{10}+1)} = \int \frac{(x^{10}+1) - x^{10}}{x(x^{10}+1)} dx$$

法2
$$\int \frac{dx}{x(x^{10}+1)} = \frac{1}{10} \int \frac{dx^{10}}{x^{10}(x^{10}+1)}$$

法3
$$\int \frac{dx}{x(x^{10}+1)} = \int \frac{dx}{x^{11}(1+x^{-10})} = \frac{-1}{10} \int \frac{dx^{-10}}{1+x^{-10}}$$



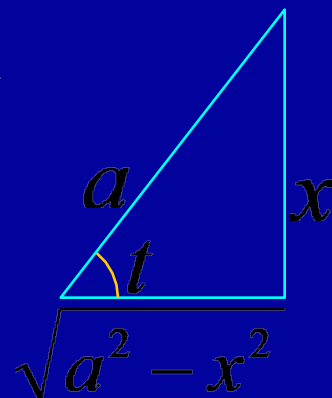
例16. 求 $\int \sqrt{a^2 - x^2} dx$ ($a > 0$).

解: 令 $x = a \sin t$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, 则

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 t} = a \cos t$$

$$dx = a \cos t dt$$

$$\therefore \text{原式} = \int a \cos t \cdot a \cos t dt = a^2 \int \cos^2 t dt$$



$$= a^2 \left(\frac{t}{2} + \frac{\sin 2t}{4} \right) + C$$

$$\sin 2t = 2 \sin t \cos t = 2 \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a}$$

$$= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} x \sqrt{a^2 - x^2} + C$$

(切记变量还原)



例17. 求 $\int \frac{dx}{\sqrt{x^2 + a^2}} \quad (a > 0)$.

解: 令 $x = a \tan t, t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, 则

$$\sqrt{x^2 + a^2} = \sqrt{a^2 \tan^2 t + a^2} = a \sec t$$

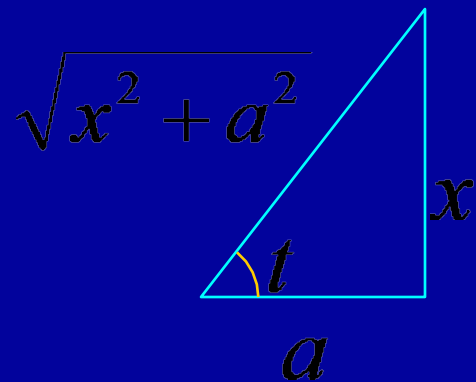
$$dx = a \sec^2 t dt$$

$$\therefore \text{原式} = \int \frac{a \sec^2 t}{a \sec t} dt = \int \sec t dt$$

$$= \ln |\sec t + \tan t| + C_1$$

$$= \ln \left[\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right] + C_1$$

$$= \ln \left[x + \sqrt{x^2 + a^2} \right] + C \quad (C = C_1 - \ln a)$$



例18. 求 $\int \frac{dx}{\sqrt{x^2 - a^2}} \quad (a > 0)$.

解: 当 $x > a$ 时, 令 $x = a \sec t, t \in (0, \frac{\pi}{2})$, 则

$$\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 t - a^2} = a \tan t$$

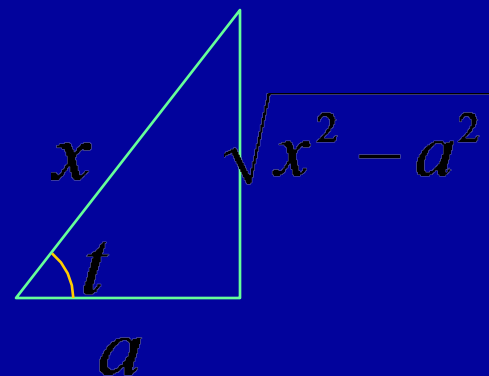
$$dx = a \sec t \tan t dt$$

$$\therefore \text{原式} = \int \frac{a \sec t \tan t}{a \tan t} dt = \int \sec t dt$$

$$= \ln |\sec t + \tan t| + C_1$$

$$= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C_1$$

$$= \ln |x + \sqrt{x^2 - a^2}| + C \quad (C = C_1 - \ln a)$$



当 $x < -a$ 时, 令 $x = -u$ 则 $u > a$, 于是

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 - a^2}} &= -\int \frac{du}{\sqrt{u^2 - a^2}} = -\ln \left| u + \sqrt{u^2 - a^2} \right| + C_1 \\ &= -\ln \left| -x + \sqrt{x^2 - a^2} \right| + C_1 \\ &= -\ln \left| \frac{a^2}{-x - \sqrt{x^2 - a^2}} \right| + C_1 \\ &= \ln \left| x + \sqrt{x^2 - a^2} \right| + C \quad (C = C_1 - 2 \ln a)\end{aligned}$$

$x > a$ 时, $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + C$



例19. 求 $\int \frac{\sqrt{a^2 - x^2}}{x^4} dx$.

解: 令 $x = \frac{1}{t}$, 则 $dx = \frac{-1}{t^2} dt$ (倒代换)

$$\text{原式} = \int \frac{\sqrt{a^2 - \frac{1}{t^2}}}{\frac{1}{t^4}} \cdot \frac{-1}{t^2} dt = -\int (a^2 t^2 - 1)^{\frac{1}{2}} |t| dt$$

当 $x > 0$ 时,

$$\begin{aligned} \text{原式} &= -\frac{1}{2a^2} \int (a^2 t^2 - 1)^{\frac{1}{2}} d(a^2 t^2 - 1) \\ &= -\frac{(a^2 t^2 - 1)^{\frac{3}{2}}}{3a^2} + C = -\frac{(a^2 - x^2)^{\frac{3}{2}}}{3a^2 x^3} + C \end{aligned}$$

当 $x < 0$ 时, 类似可得一样结果.



小结:

1. 第二类换元法常见类型:

$$(1) \int f(x, \sqrt[n]{ax+b}) dx, \quad \text{令 } t = \sqrt[n]{ax+b}$$

$$(2) \int f(x, \sqrt[n]{\frac{ax+b}{cx+d}}) dx, \quad \text{令 } t = \sqrt[n]{\frac{ax+b}{cx+d}}$$

$$(3) \int f(x, \sqrt{a^2 - x^2}) dx, \quad \text{令 } x = a \sin t \quad \text{或} \quad x = a \cos t$$

$$(4) \int f(x, \sqrt{a^2 + x^2}) dx, \quad \text{令 } x = a \tan t \quad \text{或} \quad x = a \cot t$$

$$(5) \int f(x, \sqrt{x^2 - a^2}) dx, \quad \text{令 } x = a \sec t \quad \text{或} \quad x = a \csc t$$

第四节讲



$$(6) \int f(a^x) dx, \text{ 令 } t = a^x$$

(7) 分母中因子次数较高时, 可试用**倒代换**

2. 惯用基本积分公式补充 (P205)

$$(16) \int \tan x dx = -\ln|\cos x| + C$$

$$(17) \int \cot x dx = \ln|\sin x| + C$$

$$(18) \int \sec x dx = \ln|\sec x + \tan x| + C$$

$$(19) \int \csc x dx = \ln|\csc x - \cot x| + C$$



$$(20) \quad \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$(21) \quad \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$(22) \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$$

$$(23) \quad \int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2}) + C$$

$$(24) \quad \int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln |x + \sqrt{x^2 - a^2}| + C$$



例20. 求 $\int \frac{dx}{x^2 + 2x + 3}$.

解: 原式 $= \int \frac{1}{(x+1)^2 + (\sqrt{2})^2} d(x+1)$
 $= \frac{1}{\sqrt{2}} \arctan \frac{x+1}{\sqrt{2}} + C$ (P205公式 (20))

例21. 求 $I = \int \frac{dx}{\sqrt{4x^2 + 9}}$.

解: $I = \frac{1}{2} \int \frac{d(2x)}{\sqrt{(2x)^2 + 3^2}} = \frac{1}{2} \ln (2x + \sqrt{4x^2 + 9}) + C$
(P205公式 (23))



例22. 求 $\int \frac{dx}{\sqrt{1+x-x^2}}$.

解: 原式 $= \int \frac{d(x-\frac{1}{2})}{\sqrt{(\frac{\sqrt{5}}{2})^2 - (x-\frac{1}{2})^2}} = \arcsin \frac{2x-1}{\sqrt{5}} + C$
(P205 公式 (22))

例23. 求 $\int \frac{dx}{\sqrt{e^{2x}-1}}$.

解: 原式 $= -\int \frac{de^{-x}}{\sqrt{1-e^{-2x}}} = -\arcsin e^{-x} + C$
(P205 公式 (22))



例24. 求 $\int \frac{dx}{x^2 \sqrt{x^2 + a^2}}$.

解: 令 $x = \frac{1}{t}$, 得

$$\begin{aligned} \text{原式} &= -\int \frac{t}{\sqrt{a^2 t^2 + 1}} dt \\ &= -\frac{1}{2a^2} \int \frac{d(a^2 t^2 + 1)}{\sqrt{a^2 t^2 + 1}} = -\frac{1}{a^2} \sqrt{a^2 t^2 + 1} + C \\ &= -\frac{\sqrt{x^2 + a^2}}{a^2 x} + C \end{aligned}$$



例25. 求 $\int \frac{dx}{(x+1)^3 \sqrt{x^2+2x}}$.

解: 原式 $= \int \frac{dx}{(x+1)^3 \sqrt{(x+1)^2-1}}$

$$\text{令 } x+1 = \frac{1}{t}$$

$$= \int \frac{t^3}{\sqrt{\frac{1}{t^2}-1}} \left(-\frac{1}{t^2}\right) dt = -\int \frac{t^2}{\sqrt{1-t^2}} dt$$

$$= \int \frac{(1-t^2)-1}{\sqrt{1-t^2}} dt = \int \sqrt{1-t^2} dt - \int \frac{1}{\sqrt{1-t^2}} dt$$

$$= \frac{1}{2} t \sqrt{1-t^2} + \frac{1}{2} \arcsin t - \arcsin t + C$$

$$= \frac{1}{2} \frac{\sqrt{x^2+2x}}{(x+1)^2} - \frac{1}{2} \arcsin \frac{1}{x+1} + C$$



思索与练习

1. 以下积分应怎样换元才使积分简便？

$$(1) \int \frac{x^5}{\sqrt{1+x^2}} dx$$

$$\text{令 } t = \sqrt{1+x^2}$$

$$(2) \int \frac{dx}{\sqrt{1+e^x}}$$

$$\text{令 } t = \sqrt{1+e^x}$$

$$(3) \int \frac{dx}{x(x^7+2)}$$

$$\text{令 } t = \frac{1}{x}$$



2. 求以下积分:

$$\begin{aligned} 1) \int \frac{x^2}{\sqrt{x^3+1}} dx &= \frac{1}{3} \int \frac{1}{\sqrt{x^3+1}} d(x^3+1) \\ &= \frac{2}{3} \sqrt{x^3+1} + C \end{aligned}$$

$$\begin{aligned} 2) \int \frac{2x+3}{\sqrt{1+2x-x^2}} dx &= \int \frac{-(2-2x)+5}{\sqrt{1+2x-x^2}} dx \\ &= -\int \frac{d(1+2x-x^2)}{\sqrt{1+2x-x^2}} + 5 \int \frac{d(x-1)}{\sqrt{2-(x-1)^2}} \\ &= -2\sqrt{1+2x-x^2} + 5 \arcsin \frac{x-1}{\sqrt{2}} + C \end{aligned}$$



3. 求不定积分 $\int \frac{2 \sin x \cos x \sqrt{1 + \sin^2 x}}{2 + \sin^2 x} dx$.

解: 利用凑微分法, 得

$$\text{原式} = \int \frac{\sqrt{1 + \sin^2 x}}{2 + \sin^2 x} d(1 + \sin^2 x)$$

$$\downarrow \quad \text{令 } t = \sqrt{1 + \sin^2 x}$$

$$= \int \frac{2t^2}{1+t^2} dt = 2 \int \left(1 - \frac{1}{1+t^2}\right) dt$$

$$= 2t - 2 \arctan t + C$$

$$= 2 \left[\sqrt{1 + \sin^2 x} - \arctan \sqrt{1 + \sin^2 x} \right] + C$$



4. 求不定积分 $\int \frac{1}{(1+x^2)\sqrt{1-x^2}} dx$.

解: 令 $x = \sin t$, $1+x^2 = 1+\sin^2 t$, $dx = \cos t dt$

$$\text{原式} = \int \frac{\cos t}{(1+\sin^2 t)\cos t} dt = \int \frac{1}{1+\sin^2 t} dt$$

分子分母同除以 $\cos^2 t$

$$= \int \frac{\sec^2 t}{\sec^2 t + \tan^2 t} dt = \int \frac{1}{1+2\tan^2 t} dt$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{1+(\sqrt{2}\tan t)^2} d\sqrt{2}\tan t$$

$$= \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\tan t) + C = \frac{1}{\sqrt{2}} \arctan \frac{\sqrt{2}x}{\sqrt{1-x^2}} + C$$



例1. 求 $\int x \cos x \, dx$. $\int uv' \, dx = uv - \int u'v \, dx$

解: 令 $u = x$, $v' = \cos x$,

则 $u' = 1$, $v = \sin x$

$$\begin{aligned} \therefore \text{原式} &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C \end{aligned}$$

思索: 怎样求 $\int x \sin x \, dx$, $\int x^2 \sin x \, dx$?

提醒: 令 $u = x^2$, $v' = \sin x$. 则

$$\text{原式} = -x^2 \cos x + 2 \int x \cos x \, dx$$



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