关于多元复合函数求导法则

一、链式法则

一元复合函数

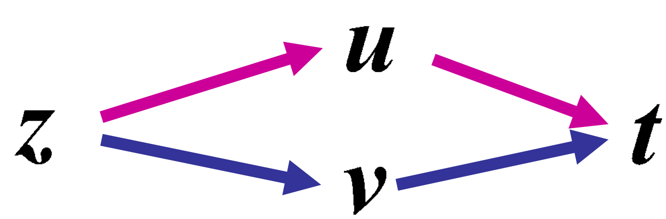
$$y = f(u),\ u = \varphi(x)$$

求导法则 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x}$

定理

如果函数 $u = \varphi(t)$ 及 $v = \psi(t)$ 都在点 t 可导，函数 $z = f(u,v)$ 在对应点 (u,v) 具有连续偏导数，则复合函数 $z = f[\varphi(t),\psi(t)]$ 在对应点 t 可导，且其导数可用下列公式计算

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}$$



证　设 t 有增量 Δt, 则 $\Delta u=\varphi(t+\Delta t)-\varphi(t)$,

$\Delta v=\psi(t+\Delta t)-\psi(t)$; 由于函数 $z=f(u,v)$ 在点 (u,v) 有连续偏导数, 故可微, 即

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+o(\rho),\ \left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\ \right)$$

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\cdot\frac{\Delta u}{\Delta t}+\frac{\partial z}{\partial v}\cdot\frac{\Delta v}{\Delta t}+\frac{o(\rho)}{\Delta t}$$

当 $\Delta t\to 0$ 时,

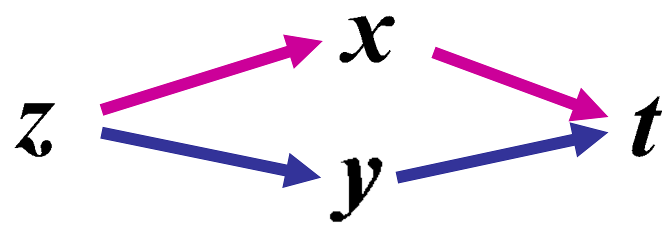
$$\frac{\Delta u}{\Delta t}\to\frac{du}{dt},\quad \frac{\Delta v}{\Delta t}\to\frac{dv}{dt},\quad \frac{o(\rho)}{\Delta t}=\frac{o(\rho)}{\rho}\left(\pm\sqrt{\left(\frac{\Delta u}{\Delta t}\right)^2+\left(\frac{\Delta v}{\Delta t}\right)^2}\right)\to 0$$

$$\therefore\ \frac{dz}{dt}=\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}.$$

$\Delta t<0$ 时, 取 “－” 号

例1 设 $z = e^{x-2y}$， 而 $x = \sin t$，$y = \varphi(t)$

其中 $\varphi(t)$ 可导，求 $\dfrac{dz}{dt}$.



解 $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= e^{x-2y} \cdot \cos t + (-2)e^{x-2y} \cdot \varphi'(t)$$

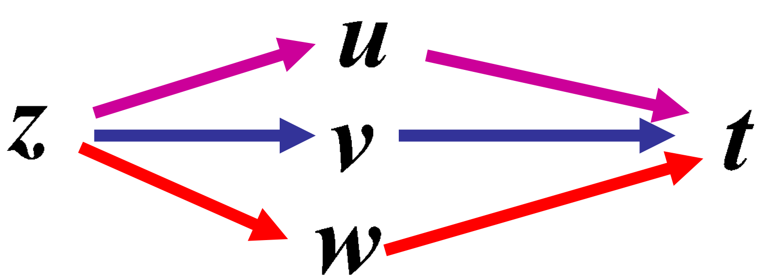
$$= e^{x-2y}[\cos t - 2\varphi'(t)]$$

推广

1. 上定理的结论可推广到

中间变量多于两个的情况：$z = f(\varphi(t), \psi(t), \omega(t))$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial w}\frac{dw}{dt}$$



以上公式中的导数 $\frac{dz}{dt}$ 称为**全导数.**

2. 上定理还可推广到中间变量不是一元函数而是多元函数的情况：

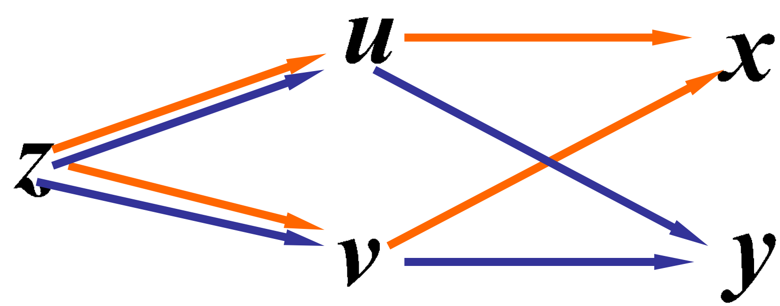
如果 $u=\varphi(x,y)$ 及 $v=\psi(x,y)$ 都在点 (x,y) 具有对 x 和 y 的偏导数，且函数 $z=f(u,v)$ 在对应点 (u,v) 具有连续偏导数，

则复合函数 $z=f[\varphi(x,y),\psi(x,y)]$ 在对应点 (x,y) 的两个偏导数存在，且可用下列公式计算：

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x},\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$z = f[\varphi(x, y), \psi(x, y)]$$

复合结构如图示



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}.$$

链式法则的规律：

“连线相乘，分线相加”

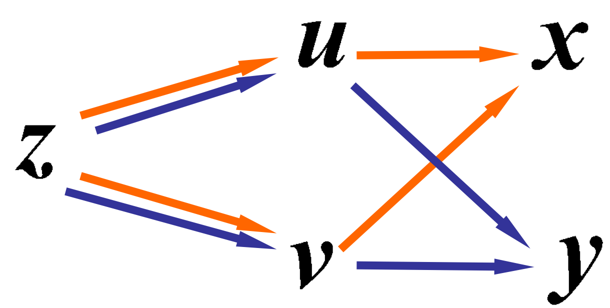
例 2 设 $z = e^u \sin v$ $u = xy$ $v = x + y$

$$\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y}$$

解 $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= e^u \sin v \cdot y + e^u \cos v \cdot 1$$

$$= e^u (y \sin v + \cos v),$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

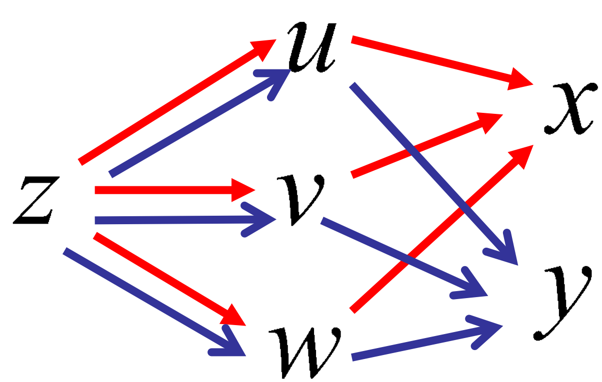
$$= e^u \sin v \cdot x + e^u \cos v \cdot 1 = e^u (x \sin v + \cos v).$$

链式法则的规律：“连线相乘，分线相加”

设 $u=\varphi(x,y), v=\psi(x,y), w=\omega(x,y)$ 都在点 (x,y) 具有偏导数，$z=f(u,v,w)$ 在对应点 (u,v,w) 具有连续偏导数，则复合函数 $z=f[\varphi(x,y),\psi(x,y),\omega(x,y)]$ 在对应点 (x,y) 的两个偏导数存在，且可用下列公式计算

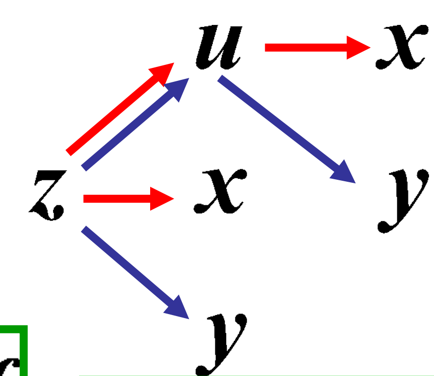
$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial y}$$



3. 中间变量即有一元函数, 也有多元函数的情况:

$z = f(u, x, y)$ 其中 $u = \varphi(x, y)$



即 $z = f[\varphi(x, y), x, y]$,

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial y}.$$

区别类似

两者的区别

把 $z = f(u, x, y)$ u y x

把复合函数 $z = f[\varphi(x, y), x, y]$

y x